Comparativism, Preferences, and the Measurement of Partial Belief

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Abstract

According to *comparativism*, an agent's beliefs consist fundamentally in a system of purely ordinal judgements of relative probability. (For example, the judgement that p is more probable than q, or that p and q are as probable as one another.) In this paper, I put forward three general challenges for comparativism, relating to its capacity to draw enough distinctions between doxastic states, its capacity to handle common instances of irrationality, and its capacity to handle interpersonal comparisons.

1. Introduction

Meet Sally. Like the rest of us, Sally has beliefs, broadly construed: there's some way she takes the world to be that's generally responsive to her evidence, which guides her intentional behaviour and the formation of her preferences. This paper concerns what Sally's beliefs might be like at the most fundamental level, and the relationship between different kinds of beliefs she seems to have.

To get the ball rolling, I'm going to assume that Sally has at least two kinds of belief: *partial* and *comparative*.[1] In the former category, for example, Sally is quite certain that there's an external world, somewhere between 95% and 98% confident that the Earth is an oblate spheroid, generally uncertain about the specific consequences of global warming, but doubtful they'll be good. These are attitudes directed towards single propositions, each of which comes with some (possibly imprecise or indeterminate) *strength*, which can (at least sometimes) be represented numerically; e.g., 'between 95% and 98% confident', 'exactly 50% confident', and so on. And with respect to her comparative beliefs, she is, for example, at least as confident that 2 and 2 makes 4 as she is anything else, she's more confident that she'll find good coffee in Melbourne than she will in Sydney, and less confident that she'll find good coffee in Sydney than that she'll

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[1] There is no well-established terminology for the distinction that I'm drawing here, so I've chosen what's easiest to write with. Alternatives would have been *absolute* versus *relative degrees of belief*; *quantitative* versus *qualitative credences*; *cardinal* versus *ordinal probabilities* (subjective); or some mixture of the above.

win the lottery next week. These are purely *ordinal* judgements of the relative probabilities of two propositions; they do not come with a strength.

Taking that assumption for granted, it's natural to wonder about the relationship between Sally's partial and comparative beliefs. It's clear enough that they're closely connected: if Sally has high confidence that p and low confidence that q, then she's more confident that p than she is that q; likewise, if Sally takes p and q to be equally likely, then she's certain that p just in case she's certain that q. These conditionals have a feel of apriority about them, and it's perfectly reasonable to think that they're underwritten by some interesting conceptual or metaphysical connection. In particular, you might think that one of the two kinds of belief is *more fundamental* than the other.

According to *comparativism*, Sally's comparative beliefs are more fundamental than her partial beliefs. Comparativism comes in a range of shapes and sizes, with roots going back to the works of de Finetti (1931) and Koopman (1940a,b); its advocates include Savage (1954), Fine (1973, 1976), Fine and Kaplan (1976), Zynda (2000), and more recently, Hawthorne (2016) and Stefánsson (2016, 2018). Roughly—I'll say more in §2—the typical comparativist might say that Sally is certain that p if and only if she considers p to be at least as probable as any other proposition whatsoever; and she's 50% confident that q whenever she takes q to be just as probable as $\neg q$.[2] One's system of beliefs is ultimately a ranking of propositions according to their relative subjective probabilities, and the different strengths of belief serve primarily as a way to represent the relative positions of propositions within that ranking.

In this paper, I raise three challenges for comparativism. Glossing over some of the details to come, I will argue that there are apparently meaningful distinctions between belief states that comparativism cannot adequately account for (§**??**); that it struggles to handle widely observed instances of irrationality (§3); and that (despite recent defences) it still lacks a plausible account of how we are able to make meaningful interpersonal comparisons of belief (§4).

2. Probabilistic Comparativism

Before we consider reasons to worry about comparativism, we should first get a clearer picture of what comparativism *is*. In fact, there are two essential components to any comparativist's theory. The first is, to put it idiomatically, an explanation of *where the numbers come from*. As B.O. Koopman once put it,

> … all the axiomatic treatments of intuitive probability current in the literature take as their starting point a number (usually between 0 and 1) corresponding to the 'degree of rational belief' or 'credibility' of the eventuality in question. Now we hold that such a number is in no wise a self-evident concomitant with or expression of the primordial intuition of probability, but rather a mathematical construct derived from [comparative beliefs] under very special conditions… (1940a, p. 269)

[2] I emphasise: comparativism, as I'm understanding it here, is *not* the view that partial beliefs depend on *some comparative thing or other*. (That's not an especially interesting thesis.) It is specifically about the relationship between partial and comparative beliefs. Comparativism can be—and often is—divorced from the thesis that beliefs depend on preferences, and the reader should be careful to keep these ideas separate for what follows.

The second component is an explanation of how those numbers come to encode *cardinal* information—how it is, for example, that ratios and intervals of numerically represented strengths of belief happen to be meaningful.

In §2.1, I'll describe the first component; and in §2.2, the second. I note, though, that I will not try to describe every possible variety of comparativism, nor even all those currently on the market. Rather, I will focus my exposition on a relatively straightforward and especially common version of the view, what I will call *probabilistic comparativism*. I will discuss how my critical arguments generalise to other types of comparativism when the appropriate time comes.

2.1 Where the numbers come from

Suppose that Sally accepts *probabilism* (as I think she should). Just what is it, then, that she accepts? According to the usual gloss, probabilism says that her beliefs ought to conform to the axioms of the probability calculus. But that can't be quite right: the axioms of the probability calculus are constraints on real-valued functions, so Sally's beliefs (whether partial or comparative) aren't even the kinds of things that might conform to them. Of course, there's no real problem here—not yet, anyway. The usual gloss was only meant to be elliptical. What was really intended is that Sally's beliefs ought to be such that they can be *represented* by a probability function.

We can break this claim down into two parts. The first is a constraint on what Sally has beliefs about; specifically, if $\mathbf{B}$ is the set of all propositions regarding which Sally has beliefs in some form or another, then $\mathbf{B}$ ought to have a 'Boolean' structure. The most general way to describe that structure is to suppose that propositions are (or can be modelled by) sets of possible worlds. We let Ω denote the set of all such worlds, and $\varnothing$ the empty set; then, probabilism requires that for any propositions p and q (i.e., any subsets of Ω),

- A1. Ω is in $\mathbf{B}$.
- A2. If p is in $\mathbf{B}$, then the complement of p with respect to Ω is also in $\mathbf{B}$.
- A3. If p and q are in $\mathbf{B}$, then the union of p and q (i.e., $p \cup q$) is also in $\mathbf{B}$.

The second part is a constraint on the beliefs themselves. It says that Sally's beliefs ought to be such that there is a function $\mathcal{P}$ from the propositions in $\mathbf{B}$ to real numbers that represents Sally's beliefs, and which satisfies the conditions:

- B1. $\mathcal{P}(\Omega) = 1$.
- B2. $\mathcal{P}(\varnothing) = 0$.
- B3. If p and q are disjoint (i.e., $p \cap q = \varnothing$), then $\mathcal{P}(p \cup q) = \mathcal{P}(p) + \mathcal{P}(q)$.

That tells us a little more about what probabilism requires, but there's still something missing: we don't yet know what it is for a real-valued function $\mathcal{P}$ to *represent* Sally's beliefs.

Probabilistic comparativism offers an answer—it's not the only possible answer, but it's not an intrinsically implausible one either, and historically it has been extremely influential. The probabilistic comparativist says that the facts about Sally's partial beliefs *supervene on*, and indeed *hold in virtue of*, the facts about her comparative beliefs; the former are reducible to the latter. Thus, a probability function *represents* Sally's beliefs when the order of the numbers it assigns to the propositions she believes corresponds to the ordering induced over those propositions by her comparative beliefs.

We can put that in more formal terms as follows. First, assume that **B** has the aforementioned Boolean structure; and second, assume that Sally's full range of comparative beliefs can be modelled with a single binary relation $\succsim$ defined over the propositions in **B**, where

$$p \succsim q \text{ iff Sally believes } p \text{ at least as much as she believes } q.$$

(I'll refer to $\succsim$ as Sally's *belief ranking*.) Where $\succ$ and $\sim$ stand for the comparatives *more probable* and *equally probable* respectively, we are in effect assuming that $p \succ q$ iff $(p \succsim q) \,\&\, \neg(q \succsim p)$, and $p \sim q$ iff $(p \succsim q) \,\&\, (q \succsim p)$. These are non-trivial assumptions, about which some comparativists might (and in fact do) disagree. But they're also assumptions that comparativists themselves frequently make, and they will help to simplify some of the technical details considerably. My critical arguments will not depend much on them, except in the precise way that they're formulated.

Given all that, the probabilistic comparativist says that probability function $\mathcal{P}$ *agrees with* Sally's belief ranking just in case

$$p \succsim q \;\text{ iff }\; \mathcal{P}(p) \geq \mathcal{P}(q).$$

Thus, if we suppose that Sally's beliefs are ultimately just her comparative beliefs, we have an unambiguous sense in which those beliefs can be represented with a probability function—i.e., a probability function represents her beliefs whenever it agrees with her belief ranking. It also gives a clear meaning to Koopman's assertion that the numbers we use to represent strengths of belief are just 'mathematical constructs' designed to help us reason about what is ultimately a system of purely ordinal judgements of relative probability. We can still say that Sally *has* partial beliefs with numerical strengths—if we want to—so long as we recognise that these are really just theoretical posits that help us to reason about where different propositions sit within Sally's belief ranking.

If we accept all this, then we can also provide an alternative formulation of exactly what probabilism requires. Since (de Finetti 1931), we've known that for probabilistic agreement it is necessary that Sally's belief ranking satisfies:

C1. $\succsim$ is transitive and complete.
C2. $\Omega \succ \varnothing$.
C3. $p \succsim \varnothing$.
C4. If $p \cap r = \varnothing$ and $q \cap r = \varnothing$, then $p \succsim q$ iff $(p \cup r) \succsim (q \cup r)$.

Conditions C2 through C4 are all eminently plausible requirements of rationality, in at least the idealised sense that probabilists tend to be interested in. (I say more about this in §3.2.) The 'completeness' part of condition C1 is a little more problematic, but we will ignore the complications it poses here.[3]

We need something a little stronger than C4 if we want necessary *and* sufficient conditions (see Kraft et al. 1959; Scott 1964, for details), and something

[3] See (Hawthorne 2016), (Seidenfeld et al. 1995), and (Alon and Lehrer 2014) for some discussion of the problems with completeness and their standard solution, which involves representations using sets of probability functions.

stronger still if we want there to be only *one* probability function that agrees with the belief ranking (see Fishburn 1986; Krantz et al. 1971, §5.2). But the details of those further (and somewhat more complicated) conditions need not concern us here, and C1–C4 are more than enough to get a handle on the kinds of constraints a belief ranking has to satisfy in order to be probabilistically representable.

2.2 Where the cardinality comes from

You might worry that something is still missing from the position that I've just been describing. In particular, the numbers we use to describe the strengths of our beliefs encode more than merely *ordinal* information. It makes sense, for example, to say that Sally might believe one thing *much more* than she believes another, or that she might believe that thing, say, *twice* as much as she believes the other. Call this *cardinal* information. What probabilistic comparativists need is an explanation of how it is that the numbers we use to represent the strengths of our beliefs manage to carry cardinal information, not just ordinal information.

In their (2011), Meacham and Weisberg claim that comparativism renders cardinal information meaningless: 'it makes sense to say that an agent regards p as more likely than q, but it does not make sense to say that she thinks p is *much* more likely than q' (p. 659). They don't give their reasons for saying this, but perhaps we can get an initial sense of a problem if we consider:

Ordinal. Sally believes p more than she believes q.

Cardinal. Sally believes p twice as much as she believes q.

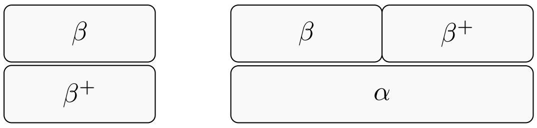
The meaningfulness of Cardinal presents a *prima facie* problem for comparativism. Cardinal implies Ordinal, but Ordinal doesn't imply Cardinal; so Cardinal carries strictly more information. Knowing just that Sally has more confidence in one proposition than another tells us nothing about *how much more* confidence is involved. However, since comparativism can only help itself directly to claims like Ordinal, it doesn't *seem* to have enough resources to explain the meaning of Cardinal.

Whatever the reason, though, the objection is much too quick. Fans of probabilistic comparativism have a long-standing story about how cardinality might be explained consistent with their view. It has been discussed in numerous locations, though in particular depth by Fine (1973, pp. 68ff). It is based on an analogy with the measurement of length, mass, and other extensive quantities. As such, I'll begin by showing how it's possible to construct a ratio scale for the measurement of length out of a purely ordinal system of length comparisons.

* * *

Let α and β be two concrete objects, and consider the claim 'α is twice as long as β'. Our task is to give truth conditions for this claim without presupposing any cardinal information.

Towards that end, suppose that β^+ is a third object that's exactly as long as β itself, and shares no parts with it. You might refer to β^+ as β's *length-duplicate*. Then, it's plausible that α is twice as long as β if, and only if, were you to take β and β^+ and join them end-to-end so as to create a new composite object, then the result would be exactly as long as α:

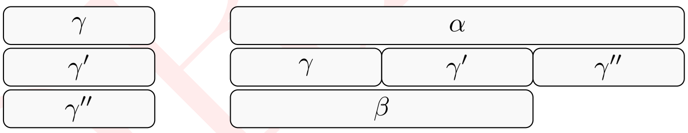


Call the object created by joining β and β^+ together in this way their *concatenation*, and denote it $\beta \oplus \beta^+$. What we've done, then, is give rough-and-ready truth conditions for our starting claim wholly in terms of purely ordinal comparisons between length-duplicates and their concatenations—i.e., the meaning of 'α is twice as long as β' decomposes into something like:

(i) β is as long as β^+, and
(ii) α is as long as $\beta \oplus \beta^+$.

(This is only a first-pass, and there are some finer points to handle before we can count it as a proper definition. For example, the act of concatenating objects will in real life very slightly alter the lengths of the component parts. But these kinds of fiddly problems are not difficult to solve, and the rough idea is all we really need to keep in mind here.)

So that's the basic idea for how a system of ordinal length comparisons can come to encode cardinal information—and it should be easy to see how it can be generalised, thus giving us a way to define up entirely arbitrary (rational) ratio comparisons. For example, suppose now that α is 1.5 times as long as β, or (equivalently) that β is two thirds as long as α. When would this be true? Well, suppose there's some third object, γ, such that α is as long as the concatenation of three length-duplicates of γ, and β is as long as the concatenation of two length-duplicates of γ. Intuitively, α is therefore three times as long as γ, and β is two times as long as γ; from there, the rest is easy.



But generalising this idea fully requires two things. The first requirement is existential: if we're going to define arbitrary ratios of lengths in terms of the concatenations of length-duplicates, then we're going to need that there are enough objects lying about with which to construct these definitions. (More precisely, we need that any time an object α is n times as long as another object β, then there had better be some γ such that α is as long as m of γ's length-duplicates, and β as long as k of γ's length-duplicates, where $m/k = n$.) This is reasonable enough in the case of length, but (as we'll see in §**??**) the analogous requirement in the case of comparativism does generate some issues.

The second requirement concerns the structural properties of the ordinal length relations themselves, and how they interact with concatenations. The very core of the definitional strategy being described here is the idea that concatenations behave as a kind of qualitative analogue of addition; that is, *the length of a concatenation is the sum of the lengths of the concatenands.* (Hence, if α is as long as the concatenation of β and β^+, then α's length is just the sum of the lengths of β and β^+, which, because they're length-duplicates, is twice the length of β.)

Once again, we can state this requirement more exactly: for the general strategy to work we need that for any non-overlapping objects α, β and γ,[4]

D1. The *is at least as long as* relation is transitive and complete.
D2. $\alpha \oplus \beta$ is at least as long as β.
D3. $\alpha \oplus \beta$ is as long as $\beta \oplus \alpha$.
D4. $\alpha \oplus (\beta \oplus \gamma)$ is as long as $(\alpha \oplus \beta) \oplus \gamma$.
D5. α is longer than β iff, for any γ, $\alpha \oplus \gamma$ is longer than $\beta \oplus \gamma$.
D6. α is not infinitely longer than β.

The reader is invited to compare D1–D6 with the essential properties of the $+$ operation in relation to $\geq$, $=$ and $>$; that is, for all non-negative reals n, m, k,

E1. $\geq$ is transitive and complete.
E2. $n + m \geq m$.
E3. $n + m = m + n$.
E4. $n + (m + k) = (n + m) + k$.
E5. $n > m$ iff, for any k, $n + k > m + k$.
E6. n is not infinitely greater than m.

One of the key results in the theory of measurement is that if the stated existential and structural requirements are satisfied, then there's a way to assign real numbers to objects such that (i) the ordinal length relations are reflected in the ordering of the numbers so assigned, and (ii) the length of a concatenation of any two non-overlapping objects is equal to the sum of their individual lengths. In other words, there's a way to assign numbers to objects—*numerical lengths*—such that those numbers encode ratio information. And once you've got meaningful ratios and sums of lengths, it's a very small step to defining various other kinds of cardinal information, e.g., differences in length, or what it is for one length to be *much* greater than another.

One quick point before we return to comparativism. The kind of story given here, of how to create a ratio scale for the measurement of a quantity, is by no means the *only* method for creating a ratio scale—it is merely the most well known. (See Krantz et al. 1971 for other methods; e.g., the method of additive conjoint measurements.) The reader should not think that, because strength of belief is measured on a ratio scale, we are therefore committed to explaining how those strengths come to carry cardinal information by some close analogy with the measurement of length.

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To show that an analogous explanation for cardinal information can *in principle* be made to work for comparative beliefs, the probabilistic comparativist needs to show that there exists some operation on propositions—the relata of a belief ranking—that might serve as a qualitative analogue of addition. In other words, comparativists need something to play the concatenation role.

Such an operation isn't too hard to find: if Sally's belief ranking agrees with a probability function, then the restriction of the union operation to disjoint

[4] The sixth, so-called 'Archimedean' condition is here stated informally. It is entirely possible to state the condition in ordinal terms, but relatively more difficult than the others. See (Krantz et al. 1971) for a thorough discussion of these conditions.

propositions will behave like addition with respect to that ranking. This is an immediate consequence of the 'additivity' axiom (B3, above) of the probability calculus: if p and q are disjoint, then the probability of their union is the sum of the probabilities of p and q separately. As a result of this fact, comparativists themselves have typically pointed towards the *union of disjoint propositions* as their proposed qualitative analogue of addition (cf. Fine 1973, p. 68; Krantz et al. 1971, p. 200; Stefánsson 2018).

But we can do better, and we should do better. The additivity axiom is a one-way conditional, and it's not true that $\mathcal{P}(p \cup q) = \mathcal{P}(p) + \mathcal{P}(q)$ *only if* p and q are disjoint. There must therefore be a strictly more general operation that's playing the concatenation role. Again, this is not too hard to find. First, say that p and q are *incompatible* (for Sally) just in case their intersection is as probable as $\varnothing$. We know that if p and q are incompatible, then $\mathcal{P}(p \cap q) = 0$ for any probability function $\mathcal{P}$ that agrees with Sally's belief ranking (if it exists). It then follows from the probability calculus that p and q are incompatible *if and only if* $\mathcal{P}(p \cup q) = \mathcal{P}(p) + \mathcal{P}(q)$. And indeed, if Sally's belief ranking does agree with a probability function, then for any *incompatible* p, q, and r,

F1. $\succsim$ is transitive and complete.
F2. $(p \cup q) \succsim q$.
F3. $(p \cup q) \sim (q \cup p)$.
F4. $(p \cup (q \cup r)) \sim ((p \cup q) \cup r)$.
F5. $p \succ q$ iff $(p \cup r) \succ (q \cup r)$.
F6. p is not infinitely more probable than q.

The upshot: if Sally's belief ranking agrees with a probability function $\mathcal{P}$, then (i) that ranking will have a well-defined 'additive' structure with respect to the union of incompatible propositions, and (ii) the numbers assigned by $\mathcal{P}$ will encode that structure in the natural way—*viz.*, the probability assigned to the union of two incompatible propositions will be the sum of the probabilities assigned to those propositions individually. So, for example, if p is as probable as the union of two incompatible propositions q_1 and q_2, each of which is as likely as the other, then p is twice as probable as q_1 (and q_2).

There is a direct analogy here with the measurement of length, and the closeness of that analogy lends considerable plausibility to the proposal. But we should be careful not to overstate what has so far been established. We have not yet been provided with a sustained argument that an agent like Sally considers p to be twice as probable as q_1 if *or* only if $p \sim (q_1 \cup q_2)$, where q_1 and q_2 are incompatible and equiprobable—even given the assumption that Sally's belief ranking agrees with a probability function. Still less have we been provided with any reasonably thorough treatment of where cardinality comes from in the event that Sally's belief ranking doesn't agree with any probability function.

Rather, what we've been given is a *how possibly* story, an account of how a comparativist *might* explain how cardinality gets encoded into numerical (specifically: probabilistic) representations of belief, under the assumption of idealised rationality. The question remains as to whether that story is correct.
In this section, I'll argue that there are apparently meaningful distinctions between belief states that probabilistic comparativism cannot account for. (§§2.3–2.6). Following that, I'll discuss how the points made against probabilistic comparativism also raise issues for comparativism more generally (§2.7).

2.3 Almost omniscience and granularity of probabilities

Suppose first of all that Sally is *almost omniscient*:

> **Example 1.** Sally is ideally rational, and her comparative beliefs satisfy all the requirements for agreement with a probability function. Furthermore, she's almost omniscient in the sense that she's narrowed down which possible world she inhabits to exactly two possibilities: either w_1 or w_2. While she's got some confidence in each, she's just a little more confident that the actual world is ω_1 than that it's ω_2.

The notion of *almost omniscience* should make sense; in fact, it already exists in the philosophical literature in the case of David Lewis' two gods (Lewis 1979, pp. 520–1). And we could easily imagine each one of Lewis' gods being more or less confident regarding which of the two (centred) worlds they inhabit by some small amount, even if the exact amount is itself imprecise to some degree. (The point here won't hinge on whether the strengths of Sally's beliefs are precise, so long as they're not radically imprecise.) So I take it that the situation described in Example 1 is conceptually possible. However, probabilistic comparativism doesn't have the resources to accommodate that possibility.

Let $\mathbf{B}$ be any set of propositions with a Boolean structure defined over any space of worlds Ω. Then, a probability function $\mathcal{P}$ agrees with Sally's belief ranking if and only if, for all propositions p in $\mathbf{B}$,

$$\mathcal{P}(p) = \begin{cases} 0, & \text{if neither } \omega_1 \text{ nor } \omega_2 \text{ are in } p, \\ x, & \text{if } \omega_1 \text{ is in } p, \text{ but } \omega_2 \text{ is not in } p, \\ y, & \text{if } \omega_2 \text{ is in } p, \text{ but } \omega_1 \text{ is not in } p, \\ 1, & \text{if both } \omega_1 \text{ and } \omega_2 \text{ are in } p, \end{cases} \qquad \text{where } 1 > x > 0.5 > y > 0.$$

This is an extremely simple relational structure, and there are uncountably many probability functions that agree with Sally' comparative beliefs (one for every real number between 0.5 and 1). So there will be one probability function where $x = 3/5$ and $y = 2/5$, for example, which perhaps fits with the description that Sally takes ω_1 to be *just a little* more probable than ω_2; but there will also be others functions where $x = 9/10$ and $y = 1/10$, or $x = 999/1000$ and $y = 1/1000$, and so on—functions where the probabilities she attaches to ω_1 and ω_2 are (almost) as far apart as they possibly can be.

Probabilistic comparativism is committed to saying that there's nothing to Sally's beliefs over and above her comparative beliefs, so any numerical differences between these probability functions cannot be meaningful. There is, therefore, no fact of the matter, or at least no determinate fact of the matter, as to *how much* more confident Sally is that the actual world is ω_1 than that it's ω_2, nor even whether she's just a little bit more confident in the former than in the latter. By virtue of learning almost everything there is to know, Sally has lost the capacity to believe one just a little more than another.

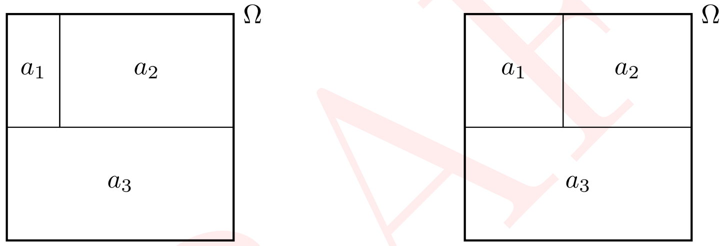
2.4 Sally and Billy

Nothing about Example 1 really hinges on Sally having narrowed things down to merely *two* possibilities, nor that those possibilities are as incredibly fine-grained as *possible worlds*. We could have made the essentially the same point given any finite number of possibilities, including possibilities that are much more coarse-grained than possible worlds.

For example, say that a proposition p is an *atom* for an agent just in case that agent has no beliefs regarding any propositions more fine-grained than p. If $\mathbf{B}$ is Boolean, then any atom must be disjoint from any other atom. Presumably, the atoms (if there are any) for any ordinary agent will be quite coarse-grained, relative to the maximal specificity of a possible world. So, now consider:

> **Example 2.** Sally and Billy have identical comparative beliefs, with respect to the same Boolean set $\mathbf{B}$ with atoms a_1, a_2, a_3. Their belief rankings agree with a probability function. Furthermore, they agree that a_3 is more likely than a_2, a_2 more likely than a_1, and $a_1 \cup a_2$ is just as likely as a_3. However, Sally insists that, unlike Billy, she has at least twice the confidence in a_2 as in a_1.

Sally and Billy's situation should again be conceivable, once the details have been worked out. In pictorial form, where the size of the boxes represent the relevant strengths of belief,



Sally Billy

But probabilistic comparativists *cannot* accept Sally's claim: if Sally and Billy have identical comparative beliefs, then they have identical beliefs *simpliciter*. (Recall the characterisation of probabilistic comparativism from §2.1.) What seems like a meaningful distinction between them is in fact meaningless.

But I think we can go beyond simple conceivability intuitions here, since we can have (theoretically well-motivated) reasons for saying that Sally attaches more confidence to a_1 than Billy does. A probability function $\mathcal{P}$ agrees with Sally's belief ranking (and hence Billy's belief ranking) just in case:

1. $\mathcal{P}(a_3) = 0.5 > \mathcal{P}(a_2) > 0.25 > \mathcal{P}(a_1) > 0$, and
2. $\mathcal{P}(a_1) + \mathcal{P}(a_2) = 0.5$.

As with Example 1, there is a multitude of probability functions that satisfy these conditions. And, crucially, *each* such function predicts a different set of preferences when it's (i) taken to model a possible belief state, and (ii) combined with any standard model of rational preference formation, e.g., expected utility theory. Suppose for instance that Sally and Billy each face choices which have the same decision-theoretic structure:

	a_1	a_2	a_3
Option α	$-2x$	x	x
Option β	0	0	x

Suppose that Sally prefers α to β; Billy has the opposite preferences. (I'll assume without argument that these preferences are possible, as they almost certainly are.) Assuming that $\mathcal{P}$ is a probability function satisfying the aforementioned conditions, then the expected utility of α is greater than the expected utility of β just in case $\mathcal{P}(a_2) > 1/3$. So, a natural explanation for the difference between their preferences would that Sally believes a_2 to a degree greater than $1/3$, and Billy less than $1/3$.

Probabilistic comparativists are committed to saying that the difference in Sally's and Billy's preferences *cannot* be due to any differences in their beliefs—there are none. Indeed, since their preferences differ but their choices have the same relevant structure, probabilistic comparativists seem committed to saying that (i) either Sally or Billy (or both) has made an irrational choice, or (ii) neither of them ought to have chosen either of the options.

Advocating the former seems an uphill battle, as there's nothing apparently *irrational* about either pattern of preferences—neither leads to Dutch bookability, and both are straightforwardly consistent with any plausible set of coherence conditions on preferences that decision theorists have laid down over the past century or so. And advocating the latter would leave us without a satisfying explanation of Sally's and Billy's preferences in the event that those preferences *consistently* pointed towards Sally believing a_2 more than $1/3$, and Billy less than $1/3$. For example, we could imagine that in the following kind of choice situation, Sally prefers δ to γ, and Billy γ to δ:

	a_1	a_2	a_3
γ	$2x$	$-x$	x
δ	0	0	x

It would be simple to generate countless more decision situations that might tell them apart. There's a natural explanation for why Sally and Billy would consistently choose 'as if' they believed a_2 more than $1/3$ and less than $1/3$ respectively, and it's strictly off-limits to probabilistic comparativism.

The foregoing examples suggest that belief rankings are too coarse-grained: there are conceptual possibilities that probabilistic comparativism cannot tell apart, and (moreover) our best theories of rational decision making treat those differences as theoretically relevant. The examples rest on the fact that relative to any set of propositions, any given probability function agrees with exactly one belief ranking, yet there are always some belief rankings that agree with multiple probability functions; and generally speaking, *any* time we have two ordinally equivalent probability functions, the numerical differences between them will be relevant for some choice situation. For the same reason, those differences will be required to appropriately rationalise some perfectly coherent set of preferences.[5] So the claim that ordinally equivalent but numerically distinct probability func-

[5] An additional (albeit minor) issue arises if $\mathcal{P}$ and $\mathcal{P}^\star$ might represent distinct distributions of *chances*. Given the Principal Principle (Lewis 1980), we could be faced with an impossible task: to have beliefs that match the chances, even where the chances cut more finely than the beliefs possibly can. But it is unclear whether the relevant kind of case can exist—to the best of my awareness, the prospects for a "comparativism about chances" have nowhere been discussed in the philosophical literature, and the plausibility of such a view will obviously depend on our theory of what chances *are*. A central issue will be whether the best theory of chance requires the richer expressive resources that come with probability functions, or whether it can make do with merely ordinal chances. I suspect it's the latter, but that is a major topic that goes well beyond what can be discussed in this paper. We do know this,

tions represent the very same state of belief is a tough pill to swallow. But it gets worse.

2.5 Radical Pyrrhonianism

Another example; this time we assume that only one probability function agrees with Sally's belief ranking:

> **Example 3.** Although her belief ranking agrees with only one probability function, Sally is *not* an ideal Bayesian agent. After reading a little too much radical Pyrrhonian literature, she insists that it's never rational to be *fully* certain of anything: one should always reserve *some* slight doubt (say, 1%) that even the most firm of logical truths might be false, and that any logical falsehood might end up true. Moreover, her preferences consistently reflect her new-found Pyrrhonian commitments—for instance, she'd prefer being given \$90 outright to the gamble $\langle \$100 \textit{ if } p \vee \neg p,\ -\$1000 \textit{ otherwise}\rangle$, and she prefers $\langle \$100 \textit{ if } p \vee \neg p,\ \$100,000 \textit{ otherwise}\rangle$ to being given \$1000 outright.

Here's one potential explanation of Sally's preferences: where $\mathcal{P}$ is the unique probability function that agrees with Sally's belief ranking, her actual partial beliefs are in fact modelled by the *non*-probabilistic function $\mathcal{P}^\star$, where

$$\mathcal{P}^\star(p) = 0.98 \cdot \mathcal{P}(p) + 0.01$$

Think of $\mathcal{P}^\star$ as $\mathcal{P}$ squished down by 1% on either side, keeping the order of the propositions fixed. How would this help to explain Sally's Pyrrhonian preferences? Because $\mathcal{P}^\star$ fits perfectly well with those preferences. From up on the high horse of probabilism we may well want to say that Sally is epistemically irrational, since she fails to attach complete certainty to self-evident tautologies. But, she's at least rational enough to choose appropriately given her slightly misled beliefs.

Such an explanation is off-limits for probabilistic comparativism. $\mathcal{P}$ and $\mathcal{P}^\star$ are ordinally equivalent, so the probabilistic comparativist is committed to saying that (at least) $\mathcal{P}$ represents Sally's beliefs. And any agent with beliefs represented by $\mathcal{P}$ obviously ought to prefer $\langle \$100 \textit{ if } p \vee \neg p,\ -\$1000 \textit{ otherwise}\rangle$ to being given \$90 outright, and prefer being given \$1000 outright to $\langle \$100 \textit{ if } p \vee \neg p,\ \$100,000 \textit{ otherwise}\rangle$. Consequently, Sally must (for *some* reason) have chosen irrationally given her beliefs. Her failing is entirely pragmatic.

That's hardly satisfying, though. We clearly have to posit some degree of irrationality somewhere, in order to make sense of Sally's non-Bayesian preferences. But the former explanation still leaves us with an agent that makes *sense*. It's easy to imagine an agent so committed to radical Pyrrhonianism that they doubt even the most obvious logical truths, being absolutely certain of nothing, and likewise give some positive probability to even the most absurd of contradictions. The latter explanation leaves us with an agent whose preferences are bafflingly nonsensical given what she supposedly believes.

though: if comparativism about chances is false, then the comparativist's owe us a new account of how chances and beliefs are supposed to interact.

Before I move on, let me just note that my point here does not commit the error that Joyce (2015, pp. 418–9) discusses in his defence against a related objection to comparativism—the error of re-scaling the model of Sally's beliefs without making appropriate adjustments to how expected utilities are calculated, thus giving the misleading impression that $\mathcal{P}^\star$ generates different predictions about Sally's preferences when they're plugged into an expected utility model of preference formation. I agree that this would be an error. But, we get to say that $\mathcal{P}^\star$ is a mere *re-scaling* of $\mathcal{P}$ only under the substantive assumption that whatever Sally believes most (or least) of all, she believes to the most (least) extent possible.

The scale we use to measure belief is a matter of stipulation, we can all agree on that. Except for making some of the mathematics easier, it doesn't really matter if we represent Sally's partial beliefs on a 0-to-1 scale, a 0.01-to-0.99 scale, or a 1-to-π scale. What we don't get to stipulate is that *being at least as confident that p as anything else* equates to *being certain that p*, and that's part of what's at issue here. When we read $\mathcal{P}^\star$, we are supposed to read it as a representation of Sally's beliefs on the *same* 0-to-1 scale that $\mathcal{P}$ uses—she's just *less than certain* of the propositions towards which she has *the most confidence*. The probabilistic comparativist has no way of expressing this possibility by appeal to ordinal comparisons between pairs of propositions.

2.6 The granularity of numerical representations

The foregoing examples are fanciful, but it's important not to miss the more general point they're intended to convey. If it sounds too strange to say that Sally can be almost omniscient, or that she might have some slight doubt in even the most basic of logical truths, then never fear: whenever any probability function agrees with a belief ranking, *no matter how uniquely it does so*, there are guaranteed to be infinitely many order-preserving transformations of that function (all with values bounded by 1 and 0) that also agree with that belief ranking, which we could draw upon to find counterexamples instead.

Perhaps the probabilistic comparativists could argue that *some* pairs of ordinally-equivalent functions will fail to pick out genuinely distinct belief states, but they will be hard pressed to show this for *all* such pairs. Consider, for instance:

> **Example 4.** After reading Descartes' *Meditations*, Sally converts to a different kind of scepticism: she now doubts every contingent claim, and is certain of only the self-evident logical truths. Her comparative beliefs, it turns out, agree with exactly one probability function, $\mathcal{P}$. But her own self-attributions and preferences fit better with:
>
> $$\mathcal{P}^\dagger(p) = \begin{cases} {}^1\!/_2 \times \mathcal{P}(p), & \text{whenever } \mathcal{P}(p) < 1 \\ \mathcal{P}(p) & \text{otherwise.} \end{cases}$$

$\mathcal{P}^\dagger$ is ordinally equivalent to $\mathcal{P}$ but with most values bunched up in the $[0, 0.5]$ range. Having partial beliefs like these would be irrational. It may even be impossible. I'm inclined to think that it isn't, but that set *that* issue aside and compare $\mathcal{P}^\dagger$ to another ordinal transformation of $\mathcal{P}$ that more smoothly shifts some of $\mathcal{P}$'s values a little to the left:

Example 5. Sally has gotten over her sceptical days, and is now *almost* an ideal Bayesian. Her comparative beliefs, it turns out, agree with exactly one probability function, $\mathcal{P}$. But her own self-attributions and preferences fit better with:

$$\mathcal{P}^{\dagger\dagger}(p) = \begin{cases} {}^{1}\!/_{2} \times \mathcal{P}(p), & \text{whenever } 0 \leq \mathcal{P}(p) < 0.05 \\ {}^{3}\!/_{4} \times \mathcal{P}(p), & \text{whenever } 0.05 \leq \mathcal{P}(p) < 0.075 \\ {}^{7}\!/_{8} \times \mathcal{P}(p), & \text{whenever } 0.075 \leq \mathcal{P}(p) < 0.0825 \\ \vdots & \\ \mathcal{P}(p) & \text{otherwise} \end{cases}$$

Such a distribution of probabilities looks neither absurd nor impossible. But the probabilistic comparativist can't even say that $\mathcal{P}^{\dagger\dagger}$ represents a *possible* state of belief, at least not *adequately* so. For note that although $\mathcal{P}^{\dagger\dagger}$ and $\mathcal{P}$ are ordinally equivalent, they disagree with respect to ratio information. So at most one of them can adequately represent whatever cardinal structure is encoded in Sally's comparative beliefs, in the sense that was characterised back in §2.2. The probabilistic comparativist is committed to saying that $\mathcal{P}$ is adequate, so $\mathcal{P}^{\dagger\dagger}$ isn't. It doesn't even represent a *possible* state of belief.

2.7 Supplemented Comparativism

Each of the examples in this section involves non-uniqueness in some way. One lesson to draw from them is that comparative beliefs alone are too coarse-grained to make sense of distinctions between states of partial belief that seem meaningful, and which are treated as meaningful by contemporary theories of decision making. Consequently, a natural thought would be to enrich the supervenience base. If *mere* comparative beliefs aren't enough to make the relevant distinctions, then perhaps comparative beliefs plus something else will suffice.

Let *supplemented comparativism* denote the view that partial beliefs supervene on comparative beliefs *plus something else* (whatever that may be, so long as it doesn't trivialise the whole affair). For example, although there are many probability functions that agree with Sally's belief ranking in Example 1, the supplemented comparativist might argue that only one of those functions has maximal fit with her evidence, and hence it counts as the uniquely proper representation of Sally's beliefs. Likewise, if we want to generate a meaningful distinction between the probability function $\mathcal{P}$ and the non-probabilistic $\mathcal{P}^{\star}$ from Example 3, then the fact that the latter of those fits better with Sally's preferences could be built into the theory so as to allow for exactly that distinction. A representation theorem similar to that of (Joyce 1999) could help in providing the mathematical foundations for this kind of view. Or, we could perhaps supplement the supervenience base with other forms of qualitative but still doxastic judgements, like whether p and q are probabilistically independent, or whether p is evidence for q (cf. Joyce 2010, p. 288).

I think the examples of this section give substantial reasons to worry about supplemented comparativism. It is not enough to just fix on a unique assignment of numerical values (precise or imprecise) to represent Sally's beliefs in the examples presented above. The *fine-grainedness* of comparative beliefs and the *uniqueness* of their numerical representations is not the whole problem. Besides the need for providing some numerical representation of Sally's beliefs, we also need an explanation of how that representation comes to represent genuinely cardinal information about her beliefs—and supplemented comparativism cannot rely on the measurement analogy for this, at least not in anything like the form described in §2.2.

For example, any explanation of cardinality in terms of the union of incompatible propositions will not help us to make sense of Example 1—there are no such propositions to 'add'. The belief ranking in that example is too simple to encode any kind of 'additive' structure in any interesting sense. Or consider Example 2: if it really is the case that Sally believes a_2 twice as much as a_1 and Billy doesn't, then that difference cannot have anything to do with their comparative beliefs. Moreover, if the non-probabilistic functions like $\mathcal{P}^{\star}$, $\mathcal{P}^{\dagger}$, or $\mathcal{P}^{\dagger\dagger}$ really do provide good models of some possible agent's beliefs, then each and every one of them is *ipso facto* a counterexample to the proposal that an agent considers p to be n times as probable as q whenever she takes p to be as probable as the union of n incompatible propositions all as probable as q.

An assignment of numbers without an empirically plausible explanation for how those numbers manage to carry cardinal information isn't a solution to comparativism's problems, no matter how unique the assignment is. To deal with the examples in this section, a different explanation of cardinality is needed. Supplementing comparativism with further facts in the supervenience base to make distinctions where mere belief rankings can't just shifts the bump under the rug. Worse: it cuts the supplemented comparativist off from a useful analogy that is the basis for much of probabilistic comparativism's plausibility.

I suspect that when we finally do explain the cardinality present in our partial belief, we'll see that it has much to do with the relationship between partial beliefs and preferences under conditions of uncertainty, and comparative relations between union of incompatible propositions won't have much of a role at all. But that is something I've argued elsewhere (see [*manuscript*]), so let me now turn to a different class of problems for comparativism.

3. Problems of Irrationality

The functions $\mathcal{P}^{\dagger}$ and $\mathcal{P}^{\dagger\dagger}$ are examples of a non-additive *capacities* (see Choquet 1954). Capacities are a generalisation of classical probability functions that are frequently used in descriptively-oriented models of belief, models aimed at representing the beliefs of ordinary, non-idealised agents. Specifically, a capacity satisfies the conditions B1 and B2 from §2.1, but replaces B3 with the significantly weaker:

B4. If p implies q, then $\mathcal{P}(q) \geq \mathcal{P}(p)$.

With capacities, $\mathcal{P}(p \cup q)$ need not and frequently does not equal $\mathcal{P}(p) + \mathcal{P}(q)$ even when p and q are incompatible. $\mathcal{P}^{\dagger}$ and $\mathcal{P}^{\dagger\dagger}$ are special cases in that they are ordinally equivalent to a probability function, and therefore the belief ranking that they agree with satisfies each of the conditions F1–F6 from §2.2. They thus generate one kind of counterexample to the measurement analogy, inasmuch as they really do model possible non-probabilistic belief states.

But there are also a vast number of capacities that (i) seem to represent possible belief states, yet (ii) are not ordinally equivalent to any probability function, and more generally (iii) do not have the appropriate ordinal structure to support the measurement analogy. In particular, many capacities agree with a belief ranking that falsifies the all-important condition F5.

Consider, for example, a hypothetical case. (I'll discuss real-life cases shortly.)

> **Example 6.** Relative to Sally's comparative beliefs, $p_1, \ldots, p_{100}$ and $q_1, \ldots, q_{101}$ are two sequences of pairwise incompatible and equiprobable propositions, with $p_1 \sim q_1$. But due to an 'accounting error', Sally thinks that $p_1 \cup \cdots \cup p_{100}$ is as likely as r, which is also as likely as $q_1 \cup \cdots \cup q_{101}$.

There are many capacities that could agree with this kind of belief ranking. (Example: let the capacity be additive as usual with respect to the p_i, and subadditive by $^{100}/_{101}\%$ for significantly large unions of the q_i.) On the other hand, it's easy to see that the *union of incompatible propositions* cannot behave like qualitative analogue of addition in this case. For suppose that it did. Then Sally would believe r 100 times as much as p_1, and she would believe r 101 times as much as q_1. But she believes p_1 as much as she believes q_1, and she obviously doesn't believe r 1% more than itself. Hence, Sally's belief ranking doesn't have the right structure to support the measurement analogy.

Should this mean that Sally doesn't have partial beliefs with respect to the p_i and q_i, or that those partial beliefs don't carry any meaningful cardinal information? Maybe there's no fact of the matter as to whether she takes r to be 100 or 101 times as probable as p_1 and q_1? For my own part, I see no compelling reason not to take each and every capacity that's consistent with her belief ranking be taken to represent a distinct, in principle possible and fully determinate belief state with meaningful cardinal information. Each of them seems to make sense as such: we have a rough idea of what a person with such beliefs would be like—with respect to their betting behaviour and assertions of confidence, for example—so it's hardly inconceivable that such belief states are possible. But to have reasons to doubt comparativism, we don't need anything nearly that strong; it's enough just that there are some irrational yet still possible belief states that it cannot adequately account for.

But let's not rest on purely hypothetical cases. Capacities were introduced to model beliefs in part due to a wealth of evidence suggesting that both the absolute and comparative confidence judgements of ordinary agents fail to satisfy the strict requirements of probabilistic coherence. See, e.g., (Tversky and Kahneman 1982), (Yates and Carlson 1986), (Carlson and Yates 1989), (Bar-Hillel and Neter 1993), (Tversky and Koehler 1994). There are many examples to draw from here, but one of the most striking—and robust—is the conjunction fallacy:[6]

[6] See (Lu 2016) for a recent review of the empirical literature, and (Moro 2009) for discussion on the interpretation of the results. I am aware that the literature on the extent of human probabilistic irrationality is both vast and for the most part controversial, and the cited works cover only a tiny fraction of it. But let me say this: even the most committed of 'descriptive' Bayesians will typically claim no more than that probability functions are idealised models for analysing the computational problems that humans face in the domains where we should independently expect cognition to be especially well optimised; e.g., vision, motor control, and language processing (e.g., Griffiths et al. 2012). Outside these domains, there's widespread agreement that the evidence that we are not probabilistically coherent is overwhelming—and F5 is one of the empirically least plausible conditions on comparative beliefs.

p. Linda is a bank teller.

q. Linda is active in the feminist movement.

$p \cap q$. Linda is a bank teller and active in the feminist movement.

A large number of people—the exact percentage doesn't matter—when they are asked to judge the relative probabilities of these propositions, will commit the *single conjunction fallacy*: they will say that $p \cap q$ is more probable than one of the other propositions (usually p), and less probable than the other. A significant number will even commit the *double conjunction fallacy*: they will say that $p \cap q$ is more probable than both p and q.

The propositions p and q in this case are not incompatible, but it doesn't take much work to show that instances of the conjunction fallacy run up against the general proposal that the probability of the union of incompatible propositions is the sum of the probabilities of those propositions. For in this case, $p \cap q$ is (obviously) incompatible with $p \cap \neg q$, and p is just $(p \cap q) \cup (p \cap \neg q)$. Hence, for the analogy to hold, we would need:

$$\mathcal{P}(p) = \mathcal{P}\big((p \cap q) \cup (p \cap \neg q)\big) = \mathcal{P}(p \cap q) + \mathcal{P}(p \cap \neg q).$$

However, $p \cap q \succ p$, so if $\mathcal{P}$ agrees with $\succsim$, then $\mathcal{P}(p \cap q) > \mathcal{P}(p)$—which would require that $\mathcal{P}(p \cap \neg q) < 0$, a nonsensical probability assignment.

So we know that ordinary agents are probabilistically incoherent, and that their comparative belief rankings often don't have the appropriate structure to support the measurement analogy. Yet this surely doesn't prevent them from having beliefs with meaningful cardinal information. A person who falls foul of the conjunction fallacy might still believe, for example, that $p \cap q$ is *just a little* more probable than p, perhaps just 2% more probable (say); while another might believe that it's *much* more probable than p. I take it that this is obvious—or, at least, that if we're going to say otherwise, then compelling reasons would be required. The people who commit the conjunction fallacy don't lose their capacity to believe the relevant propositions with cardinally meaningful strengths. Hence, a a major (and still very much unanswered) challenge for comparativism is to explain cardinality in the face of ordinary human irrationality. In the remainder of this section, I want to briefly consider three responses to this challenge.

3.1 Impossible Worlds

My argument that the conjunction fallacy is inconsistent with a belief ranking that supports the measurement analogy relies strongly on the assumption that Ω is a set of *logically possible* worlds, closed (at least) under the classical introduction and elimination rules for negation and conjunction or disjunction. However, if Ω were to include enough logically impossible worlds of the right kind, then the argument would be invalid.

More generally, we know that any apparently non-probabilistic belief ranking with respect to propositions drawn from one set of worlds Ω can be embedded into a fully probabilistic belief ranking with respect to propositions drawn from a larger space of worlds, Ω^+. See, for example, (Cozic 2006) and (Halpern and Pucella 2011); in (Elliott forthcoming), I show that for the fully general result to hold, Ω^+ needs to include not only logically impossible worlds, but also 'incomplete' worlds—i.e., worlds that leave some matters unspecified. So perhaps comparativists might maintain the measurement analogy if they let propositions be characterised as sets of possible *and* impossible/incomplete worlds.

I have raised this objection only to acknowledge it, and then set it aside. In other works, I have argued that letting Ω include logically impossible worlds creates special problems in the probabilistic context (Elliott forthcoming), and will in fact severely *undermine* the comparativist's measurement analogy rather than support it ([*manuscript*]). I won't repeat those arguments here, but let me add: if saving the measurement analogy requires the use of logically impossible and incomplete worlds, then that is a significant theoretical cost for the view. There are general reasons to worry about the use of sets of possible *and* impossible/incomplete worlds as models of belief content (e.g., Bjerring 2013; Bjerring and Schwarz 2017), so comparativists might not want to put all their eggs into this one basket.

3.2 'I only want to model ideally rational agents'

One common response to several of the foregoing examples, all of which involve non-ideal agents with comparative and/or partial beliefs that flout the orthodox norms of probabilism, is that such agents simply aren't relevant for the purposes of philosophical theorising. The idea, I take it, is that we can safely ignore non-ideal and irrational agents, at least for now, because what matters for philosophical purposes is that we have an explanation of cardinality for *ideally rational agents*. And measurement analogy seems to work well for ideally rational agents—at least when we set aside 'extreme' cases like Example 1.

Now, there's a reason why the measurement analogy works particularly well for ideally rational agents, and it has nothing at all to do with comparativism: the claim that the probability of the union of incompatible propositions ought to equal the sum of the probabilities of those propositions individually is a good norm of rationality. According to probabilism, an agent is rational only if she assigns probabilities in a manner that fits with what's required by the measurement analogy. So there's no surprise that, given a probabilist's conception of rationality, a rational agent takes p to be twice as likely as q just in case she considers p to be as likely as the union of q and q', where $q \sim q'$ and $q \cap q \sim \varnothing$. This is common ground for comparativists and non-comparativists alike. The question is whether this fact reflects some interesting dependence relationship.

If it does, then presumably that same relationship should hold for non-ideal agents. It would be unreasonable to say that Sally doesn't have partial beliefs encoding interesting cardinal information just because she isn't ideally rational. I have partial beliefs with meaningful cardinal information, and I'm far from ideally rational. So, comparativists should be able to show that their explanation of cardinality is *generalisable*. For if there doesn't seem much hope for generalising to non-ideal agents, then we've got reason to think that the explanation is false—even in the case of ideally rational agents.

What we've seen over the past two sections is that the explanation isn't generalisable. The measurement analogy requires that (i) there's enough equiprobable and incompatible propositions around to 'add'; (ii) an assumed equivalence of maximality with certainty and minimality with zero confidence; and (iii) that the belief rankings of ordinary agents will satisfy quite strong—and empirically dubious—conditions like F5. The prospects for generalisation are not strong.

3.3 Disjunctivism

Another common response to cases involving probabilistically incoherent belief rankings is *disjunctivism*: if the union of equiprobable and incompatible propositions doesn't behave additively for some agents' belief rankings, then perhaps there will be a different operation that *does* behave additively. If so, the proposal goes, then we can explain the cardinality encoded in their beliefs in terms of *that* operation instead.

Note, first of all, that disjunctivism doesn't seem to help with any of the counterexamples discussed in §**??**, where the belief rankings *did* have the required structure. At best, it helps with cases like Example 6, and even in those cases the suggestion will not be able to avoid analogous worries arising from ordinally equivalent functions.

But, moreover, if different operations are supposed to play the concatenation role for different agents, each one contingent on whatever operation is appropriate for that agent's idiosyncratic belief ranking, then both interpersonal and intrapersonal comparisons of belief would become quite useless in general. Before we could know what it means for Sally to believe p twice as much as q, we would first have to take into account her entire belief ranking, work out what the relevant 'concatenation' operation is, and only then give some empirical meaning to the statement. Without knowledge of the overall structure of her belief ranking, such a claim would only tell us that:

(i) Sally believes p more than q, and
(ii) There's *some* binary operation $\circ$ that shares certain structural characteristics with addition relative to Sally's belief ranking such that for q', q'' that are as probable as q, $q' \circ q'' = p$.

The latter is deeply uninformative; and the former we don't need cardinal information to express. And there's no guarantee that the cardinal information we get out will track the most obvious implications of believing p twice as much as q—e.g., being more likely to bet twice as much on the former as on the latter.

The suggestion is a non-starter. Disjunctivism won't help to preserve meaningful cardinal information for irrational agents, it will obliterate any meaning that cardinality seems to have.

4. Problems of Interpersonal Comparability

A different concern now: comparativism should be able to explain *interpersonal* comparisons of strength of belief. (See Meacham and Weisberg, 2011, pp. 659–60, for a discussion on why interpersonal comparisons are theoretically important.) In his recent defence of probabilistic comparativism, Stefánsson (2016) argues that comparativists can explain interpersonal comparisons as follows:

> It is generally assumed that… subjective probabilities (which represent strengths of belief) are interpersonally comparable… The crucial difference between desires and beliefs in this regard is the widely held assumption that any two rational people believe equally strongly whatever they fully believe (such as a tautology), and, similarly, believe equally strongly whatever they believe least of all… (pp. 81–2)

The argument proceeds: Suppose that there are unique probability functions $\mathcal{P}_A$ and $\mathcal{P}_B$ that agree with Ann's and Bob's belief rankings respectively. (We'll assume that $\mathcal{P}_A$ and $\mathcal{P}_B$ are defined on the same domain, **B**.) Thus, according to the measurement analogy, for any propositions p and q in **B**, there's a fact of the matter as to how much less p and q are believed than the maximally probable proposition Ω. So,

> … we might compare the degree to which Ann believes p with the degree to which Bob believes q, by comparing the distance between p and the tautology [i.e., Ω] according to Ann with the distance between q and the tautology according to Bob.

Moreover,

> The result of the above comparison is the same across different numerical models of Ann's and Bob's comparative beliefs [i.e., positive similarity transformations of $\mathcal{P}_A$ and $\mathcal{P}_B$]. That is, if Ann believes p more strongly than Bob believes q according to one of these models, then the same holds according to all of these models. (p. 82)

The idea seems to be that because the probability functions $\mathcal{P}_A$ and $\mathcal{P}_B$ (i) measure Ann's and Bob's beliefs respectively, and (ii) they both do so on a 0-to-1 scale, with $\mathcal{P}_A(\varnothing) = \mathcal{P}_B(\varnothing) = 0$ and $\mathcal{P}_A(\Omega) = \mathcal{P}_B(\Omega) = 1$, that (iii) we can therefore say that $\mathcal{P}_A$ and $\mathcal{P}_B$ belong to the 'same model' of partial belief, where that's taken to imply the validity of interpersonal comparisons between them. If we applied some positive similarity transformation to, say, $\mathcal{P}_A$ but not $\mathcal{P}_B$, the result would still be an adequate measure of Ann's beliefs, but it would ruin the possibility of any interpersonal comparisons. But to ensure interpersonal comparability, we just need to ensure that the numerical representations of $\succsim_A$ and $\succsim_B$ belong to the 'same model.'[7]

Much hangs on the assumption that the minima and maxima of $\succsim_A$ and $\succsim_B$ are comparable across rational agents. It's not clear to me how widely held this really is outside of orthodox Bayesian circles, and it's much less clear how we could generalise the explanation to accommodate non-ideal agents. But set those points aside, and assume that for every agent S,

1. $\succsim_S$ agrees with a unique probability function,
2. Ω is believed to the fullest extent that S can believe anything, and
3. $\varnothing$ is believed to the least extent that S can believe anything.

The real worry here is that we've been offered no real explanation of interpersonal comparability, even given these assumptions. The foregoing points are non-sequitors, they give us no reason whatsoever to think that $\mathcal{P}_A$ and $\mathcal{P}_B$ measure comparable psychological quantities.

An analogy will help to make this clear. Imagine a universe, Δ, that's finite in extent, and consists fundamentally of spherical atoms each with some non-zero diameter and non-zero mass, with no occupiable spaces between them. The non-atomic objects of this universe are the mereological sums of contiguous atoms.

[7] In (Stefánsson 2016) this argument in terms of positive *affine* transformations. In his (2018), though, Stefánsson switches to similarity transformations, as I've done here. This won't make a difference to my discussion.

Let $\mathfrak{O}$ be set of all such objects in Δ. Included in $\mathfrak{O}$ will be two special objects: $\varnothing$, the 'empty' arrangement of atoms; and Δ itself. Assume that length is always measured along some privileged axis such that every object has a unique length, and let $\succsim^l$ and $\succsim^m$ denote the *is at least as long as* and *is at least as massive as* relations respectively.

Obviously, while they're by no means identical, $\succsim^l$ and $\succsim^m$ will be correlated in many respects, and they'll share a number of their structural properties. In fact, given the intuitively additive properties of $\succsim^l$ with respect to concatenations (§2.2) and precisely analogous properties for $\succsim^m$, it's possible to construct a pair of ratio-scale measures f_l and f_m of $\succsim^l$ and $\succsim^m$ respectively, such that for all $\alpha, \beta \in \mathfrak{O}$,

(i^l) $\alpha \succsim^l \beta$ iff $f_l(\alpha) \geq f_l(\beta)$.
(ii^l) $f_l(\varnothing) = 0$ and $f_l(\Delta) = 1$.
(iii^l) If α, β share no parts, then $f_l(\alpha \oplus o_2) = f_l(\alpha) + f_l(\beta)$.

And

(i^m) $\alpha \succsim^m \beta$ iff $f_m(\alpha) \geq f_m(\beta)$.
(ii^m) $f_m(\varnothing) = 0$ and $f_m(\Delta) = 1$.
(iii^m) If α, β share no parts, then $f_m(\alpha \oplus o_2) = f_m(\alpha) + f_m(\beta)$.

Indeed, f_l will be strictly unique, in the sense that no other function will satisfy (i^l) through (iii^l); and likewise for f_m, *mutatis mutandis*. But it should go without saying that none of this implies that lengths and masses are comparable, in the sense that if $f_l(\alpha) \geq f_m(\beta)$, then α has as at least as much length as β has mass.

Now suppose we now stipulate that Δ has as much length as it has mass. Maybe then we could derive additional length-mass comparisons for arbitrary pairs of objects; e.g., if $f_l(\alpha) = 0.5$, then α will have less length than Δ has mass, because α has less length than Δ and Δ has just as much length as it has mass, so α must have less length than Δ has mass. And furthermore, so long as we stick to making sure length and mass are being measured on 'the same model,' all such length-mass comparisons will remain unchanged. That is, if $f_l(\varnothing) = f_m(\varnothing)$, $f_l(\Delta) = f_m(\Delta)$, and $f_l(\alpha) \geq f_m(\beta)$, then those relations will be preserved under any pair of functions $f_l^\star$ and $f_m^\star$ that we define by applying the same positive similarity transformation to f_l and f_m respectively.

So we've learnt that if you stipulate that length and mass have comparable minima and/or maxima, then you'll be able to construct a privileged set of pairs of functions according to which lengths and masses can be compared more generally. But this is entirely uninteresting: the initial stipulation is false, and the resulting comparisons are wholly artificial. The fact that circumstances might conspire such that we can measure the two loosely correlated relations over the same domain with functions that share a lot of their formal properties does nothing to change the fact that length-mass comparisons are *meaningless*.

If length-mass comparisons *were* meaningful, then there would be some interesting scale-independent physical relation that holds between α and β whenever α has at least as much length as β has mass. But since nothing interesting in physics changes when we, say, hold the scale for length fixed while varying the scale for mass, there's no genuine basis for making such comparisons. Likewise, to explain interpersonal comparisons of partial belief, we need an interesting scale-independent relation that holds between Ann and Bob when Ann believes p at least as much as Bob believes q.

I doubt that we'll find any such relation merely by looking at $\succsim_A$ and $\succsim_B$. The quantities they correspond to could be as different from one another as $\succsim^l$ and $\succsim^m$. Rather, we'll likely find the proper explanation for interpersonal comparisons in the broader role that partial beliefs play in the psychologies of the agents that have them. (For example, agents with similar strengths of belief regarding some proposition p will choose in similar ways when faced with choices conditional on p, supposing that their desires are also similar.) Once again, it seems, we need to take preferences into account.

5. Conclusion

Koopman was right about this at least: the numbers we use to refer to and reason about strengths of beliefs are not essential to them. Nobody seriously thinks that there are numbers literally in the head. Rather, there must be some fundamentally qualitative psychological phenomenon whose structure those numbers somehow represent. The hard part is saying what that structure is.

Luckily for us, there are multiple approaches to explaining in purely qualitative terms how we can come to have partial beliefs with meaningful cardinal information. I prefer an approach that originates with Ramsey (1931) and centrally involves preferences: in any well-functioning human being, partial beliefs are causally tied to preferences in such a way that meaningful cardinal information can be extracted from their interactions (see [*anonymised*]; [*anonymised*]). But there are other approaches that may also work. For example, some have suggested that *comparative expectations* over random variables are basic, and from these both partial and comparative beliefs can be derived simultaneously (see, e.g., Suppes and Zanotti 1976).[8] These approaches allow for distinctions between probability functions that probabilistic comparativism cannot make, though as yet there's been no serious philosophical work done on the empirical plausibility of the view. Or, if partial beliefs are really just outright beliefs about objective probabilities, then perhaps whatever cardinality they possess is derivative upon the cardinal information possessed by those probabilities (for which a separate story would need to be told).

Whatever the right account is, I doubt that comparativists have correctly identified the actual qualitative phenomena that explain the strengths of our partial beliefs. There are apparently meaningful distinctions between belief states that comparativism cannot adequately account for; it struggles to adequately handle widely observed instances of irrationality; and it still lacks an adequate account of how we are able to make meaningful interpersonal comparisons.[9]

[8] This approach is often conflated with comparativism, but they should be kept distinct. Indeed, Suppes and Zanotti (1976) were explicitly motivated to develop the view due to the expressive limitations of comparativism. (See also Suppes 1986 for related discussion.)

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