‘Ramseyfying’ Probabilistic Comparativism

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Abstract

Comparativism is the position that the fundamental doxastic state consists in comparative beliefs (e.g., believing p to be more likely than q), with partial beliefs (e.g., believing p to degree x) being grounded in and explained by patterns amongst comparative beliefs that exist under special conditions. In this paper, I develop a version of comparativism that originates with a suggestion made by Frank Ramsey in his ‘Probability and Partial Belief’ (1929). By means of a representation theorem, I show how this ‘Ramseyan comparativism’ can be used to weaken the (unrealistically strong) conditions required for probabilistic coherence that comparativists usually rely on, while still preserving enough structure to let us retain the usual comparativists’ account of quantitative doxastic comparisons.

1 Introduction

For theorists who deal in partial beliefs, a pressing issue concerns the basis of their measurement and quantification. It is typical to represent the strengths with which propositions are believed with real numbers, or (especially in recent years) with intervals of the reals. Moreover, it’s usually taken for granted that these numerical representations encode more than merely ordinal—or: *quantitative*—information. For instance, most would be happy to treat the following inference as valid:[1]

1. α believes p to degree x
2. α believes q to degree y
3. $x = n \cdot y$

$\therefore$ α believes p n times as much as she believes q

*Email: e.j.r.elliott@leeds.ac.uk. Comments welcome; helpful comments doubly so. Do not cite without permission. Please note that this paper has nothing to do with Ramseyfication.

[1] We assume of course that p and q are being measured on the same scale.

Making sense of this kind of quantitative information is a priority in particular for *comparativism*, the position that the fundamental doxastic state consists in comparative beliefs (e.g., believing p to be more likely than q), with partial beliefs (e.g., believing p to degree x) being grounded in and explained by patterns amongst comparative beliefs that exist under special conditions.

By drawing on an analogy with the measurement of length, mass, and other basic extensive quantities, comparativists have been able to state sufficient conditions under which quantitative information can plausibly be extracted from an agent's ordinal doxastic comparisons. (See de Finetti 1931, 1937; Kraft et al. 1959; Scott 1964; Fine 1973, pp.68ff; Fishburn 1986; Stefánsson 2017.) As things stand, though, these conditions tend to be quite strong indeed—essentially imposing a qualitative form of probabilistic coherence on the agent. What comparativism currently lacks is a detailed answer as to whether, and how, the conditions can be relaxed so as to accommodate quantitative comparisons for agents more realistically construed.

This paper concerns a comparativist proposal that originates with a brief remark made in Frank Ramsey's note 'Probability and Partial Belief' (1929). I develop Ramsey's idea, using it as the basis for a representation theorem with axioms weaker than those required for probabilistic coherence. Moreover, I show how a 'Ramseyan comparativism' nicely accommodates the usual comparativist account of quantitative comparisons, by establishing more general conditions under which the purported analogy with the measurement of extensive physical quantities can be taken to hold.

I begin by introducing some key terms and assumptions in §2. I then provide an outline of the usual (probabilistic) comparativist account of quantitative comparisons in §3, before discussing the need for generalisation in §4. Finally, in §5, I develop the Ramseyan version of comparativism.

2 Basic Notation and Assumptions

I use 'α' to denote an arbitrary doxastic agent. I assume that the propositions regarding which α has beliefs can be modelled as subsets of some space of possible worlds, Ω. Furthermore, I use '$\mathfrak{B}$' to denote that set of propositions regarding which α has beliefs (i.e., whether partial or comparative). So, if α considers p to be more likely than q, then both p and q will be in $\mathfrak{B}$; and if α partially believes r to whatever degree, then r will also be in $\mathfrak{B}$. For the sake of simplicity, I assume that $\mathfrak{B}$ is a finite algebra of sets on Ω:

Definition 2.1 $\mathfrak{B}$ is a *algebra of sets on* Ω iff $\mathfrak{B} \subseteq \wp(\Omega)$, and $\forall p, q \in \wp(\Omega)$,

(i) $\Omega \in \mathfrak{B}$
(ii) If $p \in \mathfrak{B}$, then $\Omega \setminus p \in \mathfrak{B}$
(iii) If $p, q \in \mathfrak{B}$, then $p \cup q \in \mathfrak{B}$

Furthermore, a non-empty element $a \in \mathfrak{B}$ is an *atom* iff $\forall p \in \mathfrak{B}$, $a \cap p = a$ or $a \cap p = \varnothing$

I assume that α's comparative beliefs can be modelled with a single binary relation $\succsim$ on $\mathfrak{B}$, where:

$$p \succsim q \ \text{ iff } \ \alpha \text{ believes } p \text{ at least as much as she believes } q$$

I will refer to $\succsim$ as α's *belief ranking*. Implicit in this last assumption is a commitment that comparativists in general need not accept, that's worth pausing to highlight. Where $\succ$, $\prec$, $\sim$, and $\precsim$ stand for the doxastic comparatives *more*, *less*, *equally*, and *at most as much as* respectively, I am essentially assuming that the following is appropriate:

Definition 2.2 $\forall p, q \in \mathcal{B}$,
(i) $p \succ q$ iff $q \prec p$
(ii) $p \succsim q$ iff $q \precsim p$
(iii) $p \sim q$ iff $p \succsim q$ and $q \succsim p$
(iv) $p \succ q$ iff $p \succsim q$ and $q \not\succsim p$

From (iii) and (iv), it follows that $\sim$ and $\succ$ constitute the symmetric and asymmetric parts of $\succsim$ respectively; hence,

$$p \succsim q \ \text{ iff } \ p \succ q \text{ or } p \sim q$$

Nothing about Definition 2.2 should be considered obvious or trivial. For example, contrary to (iii) and (iv), α might think that p is at least as likely as q, without thereby thinking either that p is more likely than q, or that p is just as likely as q. Nevertheless, Definition 2.2 will help to simplify the following discussion considerably.

3 Quantitative Comparisons: The Usual Story

To get an initial sense of why comparativists might have troubles accounting for quantitative comparisons, contrast the purely qualitative comparison (1) with the quantitative comparisons (2) and (3):

(1) α believes p more than she believes q
(2) α believes p twice as much as she believes q
(3) α believes p much more than she believes q

Any adequate account of what our beliefs are like needs to explain the clearly sensible distinctions between these claims.[2] However, (2) and (3) present a *prima facie* problem for comparativism. Each implies (1), and in that sense carry at least as much information as is carried by the purely qualitative comparison. In the other direction, though, (1) implies neither (2) nor (3). Knowing just that α has more confidence in p than in q tells us nothing about *how much* more confidence is involved. Since comparativism can only help itself directly to

[2] The interested reader can see (Vassend forthcoming) and (Levinstein 2013, pp.23ff) for discussion on why it's important for theorists to accommodate these kinds of comparisons.

qualitative comparisons of the kind found in (1), it doesn't *seem* to have enough resources to explain quantitative comparisons.

By drawing on the theory of measurement, however, comparativists have a powerful response. We begin with an analogy, to show how it's possible to extract quantitative information about lengths from purely qualitative comparisons of relative length. I then discuss how the same strategy might be applied to beliefs.

Let o_1 and o_2 refer to a pair of concrete objects, and consider the following:

(4) o_1 is longer than o_2
(5) o_1 is twice as long as o_2
(6) o_1 is much longer than o_2

(5) obviously contains strictly more information than (4), and it's easy to see what that additional information amounts to. Suppose you were to take two objects the same length as o_2 which share no parts, and join them end-to-end; (5) then implies that the resulting object would be just as long as o_1. Roughly: o_1 is as long as two 'copies' of o_1 joined end-to-end. Call the operation of joining objects end-to-end *concatenation*. Intuitively, concatenation acts as a qualitative analogue of *adding* objects' lengths together. And once we have a way of saying what it is to 'add' lengths, it's a short step to explaining what it is for one object to be n times as long as another, or *much* longer than another. So, for (6), say that o_1 is *much longer* than o_2 just in case the difference in length between the two is at least that of some contextually-determined threshold length o_3. Then, (6) holds whenever o_1 is at least as long as o_2 concatenated with any object no longer than o_3.

Thus, we've been able to give real-world, *qualitative* meaning to the quantitative comparisons in (5) and (6) wholly by reference to properties possessed by the *is longer than* relation that it holds in connection to concatenation operations. And we can make the analogy between addition and concatenation precise. Where

$$o_1 \succsim^\star o_2 \text{ iff } o_1 \text{ is at least as long as } o_2,$$
$$o_1 \oplus o_2 = \text{the concatenation of } o_1 \text{ and } o_2,$$

it's safe to presume that $\succsim^\star$ is transitive and complete, and that $\oplus$ is *positive*, *commutative*, *associative*, and *qualitatively additive* with respect to $\succsim^\star$, in the respective senses that for all objects o_1, o_2, o_3 with non-zero length that share no parts,

(i) $o_1 \oplus o_2 \succ^\star o_1$ ($\succsim^\star$-*positivity*)
(ii) $o_1 \oplus o_2 \sim^\star o_2 \oplus o_1$ ($\succsim^\star$-*commutativity*)
(iii) $o_1 \oplus (o_2 \oplus o_3) \sim^\star (o_1 \oplus o_2) \oplus o_3$ ($\succsim^\star$-*associativity*)
(iv) $o_1 \succsim^\star o_2$ iff $o_1 \oplus o_3 \succsim^\star o_2 \oplus o_3$ ($\succsim^\star$-*qualitative additivity*)

That is: inasmuch as $\succsim^\star$ behaves like $\geq$ over the real numbers, so too does $\oplus$ behave like $+$.[3] And from this starting point, it is straightforward to develop a

[3] See (Krantz et al. 1971, §3.2.1) for further discussion. Conditions (i)-(iv) are not yet sufficient to establish that $\oplus$ can be mapped onto $+$ whenever $\succsim^\star$ is transitive and complete;

ratio-scale measure of length that can explicitly capture the quantitative structure identified in $\succsim^\star$.

So, let's return to comparativism. To establish that the same strategy can be put towards an explanation of the distinctions between (1), (2), and (3), comparativists need to identify an operation on the relata of belief rankings (i.e., sets of worlds) which behaves sufficiently like addition with respect to those rankings to justify treating it as the qualitative analogue thereof. It is common at this point to suggest the union of disjoint sets, but it's possible to say something a little more general than that. Where certain structural conditions hold true of $\succsim$, a qualitative analogue of addition exists in the union of what I'll call *epistemically exclusive* propositions, where two propositions are epistemically exclusive for α just in case she has minimal confidence in their intersection. (I define this formally below.)

To fully spell out the present suggestion, I'll need some more vocabulary. First, say that a real-valued function $\mathcal{C}r$ on $\mathbf{\mathfrak{B}}$ *agrees with* $\succsim$ just in case, for all p and q in $\mathbf{\mathfrak{B}}$,

$$p \succsim q \text{ iff } \mathcal{C}r(p) \geq \mathcal{C}r(q)$$

Say also that $\mathcal{C}r$ *almost agrees with* $\succsim$ just in case, for all p and q in $\mathbf{\mathfrak{B}}$,

$$p \succsim q \text{ only if } \mathcal{C}r(p) \geq \mathcal{C}r(q)$$

The first step is then to suppose that a *probability function* agrees with $\succsim$, where:

Definition 3.1 $\mathcal{C}r : \mathbf{\mathfrak{B}} \mapsto \mathbb{R}$ is a *probability function* iff $\mathbf{\mathfrak{B}}$ is a algebra of sets on Ω, and $\forall p, q \in \mathbf{\mathfrak{B}}$,

(i) $\mathcal{C}r(\Omega) = 1$
(ii) $\mathcal{C}r(p) \geq 0$
(iii) If $p \cap q = \varnothing$, then $\mathcal{C}r(p \cup q) = \mathcal{C}r(p) + \mathcal{C}r(q)$

Now, the conditions under which a probability function agrees with a belief ranking are well known. In the finite case, these conditions are summarised in the following theorem (due to Scott 1964):[4]

Theorem 3.1 *If $\mathbf{\mathfrak{B}}$ is finite algebra of sets on Ω and $\succsim$ is a binary relation on $\mathbf{\mathfrak{B}}$, then there is a probability function $\mathcal{C}r$ that agrees with $\succsim$ iff A1-A5 hold:*

A1. $\succsim$ *is complete*

A2. $\succsim$ *is reflexive*

A3. $\varnothing \not\succsim \Omega$

for that, an Archimedean condition is also needed: if $o_1 \succ^\star o_2$, then for any o_3, o_4, there exists a positive integer n such that $\langle n \rangle o_1 \oplus o_3 \succsim^\star \langle n \rangle o_2 \oplus o_4$, where $\langle n \rangle o_1$ is defined: $\langle 1 \rangle o_1 = o_1$, and $\langle n+1 \rangle o_1 = \langle n \rangle o_1 \oplus o_1$. Since the Archimedean condition in the probabilistic case (discussed below) is significantly more difficult to state, I've neglected to mention it here. A statement of an Archimedean condition for the additive measurement of belief rankings can be found in (Chateauneuf and Jaffray 1984, p.193).

[4] Given A1 and A5, A2 is redundant. It is included for a later discussion.

A4. $\forall p \in \mathcal{B}$, $p \succsim \varnothing$

A5. *Where $\mathbf{1}_p$ denotes the indicator function of p, and $(p_i)_{i=1}^n$ and $(q_i)_{i=1}^n$ are finite sequences of propositions from $\mathcal{B}$, then if*

(i) $\sum_{i=1}^n \mathbf{1}_{p_i}(\omega) = \sum_{i=1}^n \mathbf{1}_{q_i}(\omega)$ *for all* $\omega \in \Omega$, *and*
(ii) $p_i \succsim q_i$, *for* $i = 1, \ldots, n-1$,

then $q_n \succsim p_n$

For present purposes, the specifics of the axioms A1-A5 don't matter. What's important is what they imply with respect to epistemically exclusive propositions, which are defined as follows:[5]

Definition 3.2 $\forall p \in \mathcal{B}$,
(i) p is *minimal* iff $q \succsim p$, for all $q \in \mathcal{B}$
(ii) p is *maximal* iff $p \succsim q$, for all $q \in \mathcal{B}$

Definition 3.3 $\mathcal{P} \subseteq \mathcal{B}$ is a *set of epistemically exclusive propositions* iff, for any $\mathcal{P}' \subseteq \mathcal{P}$ s.t. $|\mathcal{P}'| \geq 2$, $\bigcap \mathcal{P}' \sim q$ for some minimal q

Definition 3.4 $p, q, \ldots$ are *epistemically exclusive* iff there is a set of epistemically exclusive propositions $\mathcal{P}$ s.t. $p, q, \ldots \in \mathcal{P}$

Assuming that if p is minimal then α has exactly zero confidence in p, Definition 3.3 plausibly characterises what it is for α to believe that at most one member of $\mathcal{P}$ can be true. In the context of A1-A5, p and q are epistemically exclusive just in case α considers their intersection to be as likely as $\varnothing$. The somewhat tortured sequence of definitions given here will be useful below, when I generalise away from probability functions.

I can now state the crucial point in relation to the quantification of belief: A1-A5 imply that $\succsim$ is transitive and complete, and that for all epistemically exclusive propositions p, q, r,

(i) $p \cup q \succsim q$, with $\succ$ replacing $\succsim$ when p is non-minimal ($\succsim$*-positivity*)
(ii) $p \cup q \sim q \cup p$ ($\succsim$*-commutativity*)
(iii) $p \cup (q \cup r) \sim (p \cup q) \cup r$ ($\succsim$*-associativity*)
(iv) $p \succsim q$ iff $p \cup r \succsim q \cup r$ ($\succsim$*-qualitative additivity*)

Furthermore, since $\mathcal{B}$ is an algebra of sets on Ω, whenever $p \subseteq q$, there will be some proposition r disjoint from (and therefore epistemically exclusive of) p such that $p \cup r = q$. Hence, it's possible to treat any non-empty proposition p in $\mathcal{B}$ as the 'sum' of some sequence of 'smaller' propositions, $p_1, \ldots, p_n$.

To turn all this into a response to the challenge with which I began this section, let *probabilistic comparativism* denote any version of comparativism committed to the following:

[5] Definition 3.3 implies that every singleton set $\{p\} \subset \wp(\Omega)$ is trivially a set of epistemically exclusive propositions. This is a feature, not a bug.

Probabilistic Comparativism: If a probability function $\mathcal{C}r$ agrees with $\succsim$, then $\mathcal{C}r$ is an adequate model of α's beliefs *simpliciter*

Note, of course, that the probabilistic comparativist is not committed to saying that partial beliefs can *only* be modelled by probability functions. This would clearly be unreasonable. For example, if all one cares about are ratios of strength of belief, then whenever a real-valued function $\mathcal{C}r$ adequately captures those ratios, so too will any positive similarity transformation of $\mathcal{C}r$. Nor should it be expected that the 'adequate' models are limited to $\mathcal{C}r$'s positive similarity transformations. (Cf. Krantz et al. 1971, §3.9, for relevant discussion.) However, it would be orthogonal to our purposes to investigate necessary and sufficient conditions for representational adequacy here, and probabilistic comparativism gives us enough to go on for now.

Taking probabilistic comparativism for granted, it's apparent how we could begin to account for quantitative comparisons. Supposing that α's comparative beliefs satisfy A1-A5, the union of epistemically exclusive propositions behaves just as one would expect of any qualitative analogue of addition. From there, we can start to cash out the meaning of quantitative belief comparisons. Consider the following:[6]

General Ratio Principle (GRP): For n, m such that $0 < n \leq m$, if there are m non-minimal epistemically exclusive propositions $r_1, \ldots, r_m$ s.t.

(i) $r_1 \sim \cdots \sim r_m$,
(ii) $r_1 \cup \cdots \cup r_m \sim q$, and
(iii) $r_1 \cup \cdots \cup r_n \sim p$,

then α believes p ${}^{n}/_{m}$ times as much as q; furthermore, if α believes p ${}^{n}/_{m}$ times as much as q, and q ${}^{n'}/_{m'}$ times as much as r, then α believes p ${}^{(n \cdot n')}/_{(m \cdot m')}$ times as much as r

So, for instance, α will take p to be twice as likely as q (and q half as likely as p) if there is some proposition q' disjoint from q such that $q \sim q'$ and $q \cup q' \sim p$.

Moreover, if a probability function $\mathcal{C}r$ almost agrees with $\succsim$, then $\mathcal{C}r$ *coheres* with the GRP, in the sense that whenever that principle implies that p is believed ${}^{n}/_{m}$ times as much as q, then

$$\mathcal{C}r(q) = {}^{n}/_{m} \cdot \mathcal{C}r(q)$$

This means it's possible to extend the account of quantitative comparisons just given into the imprecise case. This will let us weaken one of the stronger axioms mentioned in Theorem 3.1—in particular, A1, which states that $\succsim$ must be complete. For non-ideal agents (and perhaps even for ideally rational agents), completeness is widely considered implausible. Especially where $\mathcal{B}$ is very large, we should expect plenty of gaps in $\succsim$. Consider the following case, adapted from (Fishburn 1986):

[6] The first clause of the GRP is a generalisation of Stefánsson's (forthcoming) 'Ratio Principle'. The second (inductive) clause is my own addition—see §5 for a case where it's put to work.

p = The global population in 2100 will be greater than 13 billion
q = The next card drawn from this old and incomplete deck will be a heart

p and q are sufficiently far removed from one another that it's hard to make a judgement as to which is more likely than the other. Similar examples abound.

There is a natural way of dealing with incomplete belief rankings to which comparativists can appeal here. Where $\mathfrak{F}$ is any set of real-valued functions on $\mathfrak{B}$, say that $\mathfrak{F}$ agrees with $\succsim$ just in case for all $p, q \in \mathfrak{B}$,

$$p \succsim q \text{ iff } \forall \mathcal{C}r \in \mathfrak{F}, \mathcal{C}r(p) \geq \mathcal{C}r(q)$$

Modelling beliefs by sets of numerical functions works by something akin to supervaluation: only what's common to every function in $\mathfrak{F}$ is treated as having real-world import. The following theorem from (Alon and Lehrer 2014) then shows that the comparativist can do without A1 entirely:

Theorem 3.2 *If $\mathfrak{B}$ is finite algebra of sets on Ω and $\succsim$ is a binary relation on $\mathfrak{B}$, then there exists a non-empty set of probability functions $\mathfrak{F}$ that agrees with $\succsim$ iff $\succsim$ satisfies A2-A4, and*

A5⋆. *Where $(p_i)_{i=1}^n$ and $(q_i)_{i=1}^n$ are finite sequences from $\mathfrak{B}$, and $(k_i)_{i=1}^n$ is a finite sequence from $\mathbb{N}$, then if*

(i) $\sum_{i=1}^n k_i \cdot \mathbf{1}_{p_i}(\omega) = \sum_{i=1}^n k_i \cdot \mathbf{1}_{q_i}(\omega)$ *for all* $\omega \in \Omega$, *and*
(ii) $p_i \succsim q_i$, *for* $i = 1, \ldots, n-1$,

then $q_n \succsim p_n$

Given A2-A4, A5⋆ is stronger than A5 (see Harrison-Trainor et al. 2016). Note also that while there may sometimes be more than one set of probability functions $\mathfrak{F}$ that agrees with $\succsim$, the union of all such sets will itself agree with $\succsim$. So there's always a *unique* $\mathfrak{F}$ that agrees with $\succsim$ which is maximal with respect to inclusion whenever $\succsim$ satisfies A2-A5⋆.

We can use *imprecise-probabilistic comparativism* to refer to any version of comparativism committed to the following:

Imprecise-Probabilistic Comparativism: If a non-empty set of probability functions $\mathfrak{F}$ agrees with $\succsim$ and $\mathfrak{F}$ is maximal with respect to inclusion, then $\mathfrak{F}$ is an adequate model of α's beliefs *simpliciter*

The imprecise version of probabilistic comparativism does not imply probabilistic comparativism. The two positions will diverge when more than one probability function (fully) agrees with $\succsim$. Nevertheless, since any $\mathcal{C}r$ in an agreeing set $\mathfrak{F}$ will itself almost agree with $\succsim$, the imprecise-probabilistic comparativist can retain the GRP—where the notion of *coherence* is extended in the obvious way to sets of functions, if a set of probabilities $\mathfrak{F}$ agrees with $\succsim$, then $\mathfrak{F}$ coheres with the GRP.

4 The Limits of the Usual Story

We've seen that the union of epistemically exclusive propositions behaves like addition when A1-A5 (or A2-A5*) are satisfied, but those are the kinds of conditions we could only reasonably expect to be satisfied by an ideally rational agent. An ordinary agent like α probably won't satisfy all of these conditions—arguably, not even to a very close approximation. For example, consider the monotonicity property, which is a consequence of A2-A5:

$$\text{If } p \subseteq q \text{ and } p, q \in \boldsymbol{\mathcal{B}}, \text{ then } p \precsim q \qquad (\textit{monotonicity})$$

Monotonicity generates a probabilistic version of the classic problem of logical omniscience: if the worlds in Ω are closed under any consequence relation $\Rightarrow$ whatsoever, then for all $p, q \in \boldsymbol{\mathcal{B}}$,

$$\text{If } p \Rightarrow q, \text{ then } p \precsim q \qquad (\textit{logical omniscience})$$

That is, any monotonic belief ranking over a space of worlds closed under $\Rightarrow$ is necessarily coherent with respect to $\Rightarrow$. And where $\Rightarrow$ is any reasonably strong consequence relation, it is not especially plausible that $\succsim$ will be monotonic for ordinary agents.[7]

Moreover, it's clear that the GRP cannot plausibly be applied to arbitrary belief rankings. For instance, suppose that $\succsim$ includes the following, where $q \cap q'$ is minimal:

$$q \sim q' \succ p \sim q \cup q' \succ q \cap q'$$

To apply the GRP in this case is to invite absurdity: we wouldn't want to say that α believes p twice as much as q, even while $q \succ p$! Or, if that example seems unrealistic—perhaps it requires α to be a little too irrational—then there are countless others. Suppose that $p_1, \ldots, p_n$ and $q_1, \ldots, q_{n+1}$ are two sequences of epistemically exclusive propositions such that for $i, j = 1, \ldots, n$ and $k, l = 1, \ldots, n+1$, $p_i \sim p_j$, $q_k \sim q_l$, and $p_i \sim q_k$. Now suppose that

$$p_1 \cup \cdots \cup p_n \sim q_1 \cup \cdots \cup q_{n+1}$$

The GRP now implies that

1. α believes $p_1 \cup \cdots \cup p_n$ n times as much as p_1
2. α believes $q_1 \cup \cdots \cup q_{n+1}$ $n+1$ times as much as p_1
3. α believes $p_1 \cup \cdots \cup p_n$ exactly as much as $q_1 \cup \cdots \cup q_{n+1}$

Certain kinds of irrationality ruled out by axioms A1-A5/A2-A5* render the GRP inapplicable—essentially, by breaking the analogy between addition and the union of epistemically exclusive sets. In the next section, I want to investigate

[7] It might be that impossible worlds could help to make monotonicity seem more palatable. So long as we are loose enough with what we count as a 'world', it's easy enough to construct a space of worlds Ω that isn't closed under anything but the trivial consequence relation $p \Rightarrow q$ iff $p = q$. I have argued elsewhere that this approach is problematic in the probabilistic context; see (Elliott forthcoming).

just how far we can push this analogy: at what point does it thoroughly break down? But before we get to that, I'll here briefly discuss why it's important to seek a more general basis for comparativism than the axioms required for probabilistic coherence.

First, it would be unreasonable to say that α doesn't have partial beliefs merely because she's not ideally rational, or that the satisfaction of a very strong rationality condition like A5 is necessary for the meaningfulness of quantitative comparisons.[8] That would be manifestly implausible: even if she were quite highly irrational, α could still believe one proposition *much more* than she believes another, or *at least twice as much* as she believes another.[9] This should be uncontroversial—only someone caught firmly in the grips of an unrealistic picture of belief would think to deny it. Our capacity to make quantitative belief comparisons is not hostage to any presupposition of idealised rationality. And an explanation of quantitative comparisons that works *only* in the ideal case is, at best, incomplete—and at worst, no explanation at all. All else being equal, it would be better to have an account of how we make quantitative comparisons that applies equally well to the lowest common denominator.

This is *not* to deny the obvious point that it's often useful to get an explanation of some phenomenon working for an idealised model before moving on to less ideal cases. That is how science works in general, and it's exactly how we should expect things to work here. But an idealised model does real-world explanatory work only to the extent that it does not depend critically on the idealisations in question. Models have explanatory value when the conclusions we can draw from them are robust under variations to their idealising conditions; they should not break down when realism is added back in. In the present case, then, it would be useful to have some assurance that the usual comparativist account of quantitative comparisons does not depend critically on unrealistic assumptions. Comparativism needs that the basic form of that explanation can be extended to ordinary agents—else, it needs another story for how we make quantitative comparisons.

5 Unconditional Ramseyan Comparativism

The alternative basis for comparativism that I will pursue in this section and the next are inspired by the following remark in Ramsey (1929):

> [...] 'Well, I believe it to an extent ²/₃', i.e. (this at least is the most natural interpretation) 'I have the same degree of belief in it as in

[8] I am of course aware that some Bayesians are happy to accept the probabilistic representation of beliefs as descriptively adequate for real-life agents. The literature on how close ordinary humans come to being probabilistically coherent is vast, and most of it controversial. There's more here than I can hope to address, so I'll assume without further argument that most agents deviate substantially from conditions like A1-A5/A2-A5*.

[9] The point here is independent of the matter of how *precise* the partial beliefs of ordinary agents are. Even if α's beliefs were everywhere imprecise, she could still believe p at least twice as much as q.

$p \vee q$ when I think p, q, r equally likely and know that exactly one of them is true'. (p.256)

The idea is also discussed briefly by Brian Weatherson (2016, pp.223-4). However, neither Ramsey nor Weatherson go beyond this initial suggestion, and as we'll see there are a few conditions that need to be met before we can use it to ground a plausible account of partial belief.

In this paper, I stick to the letter of the quoted passage, and develop Ramsey's idea within a comparativist framework that takes *binary* comparisons as primitive. It is also possible to develop a version of the same idea within a framework where *quarternary* comparisons are primitive; i.e., where

$$p, q \succsim r, s \text{ iff } \alpha \text{ believes } p \text{ given } q \text{ at least as much as she believes } r \text{ given } s$$

However, space constraints dictate a focus on binary comparativism here.

To begin with, we will need some additional terminology:

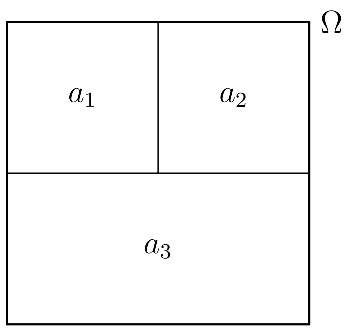
Definition 5.1 A set of n epistemically exclusive propositions $\mathcal{P}$ is an *n-scale of p* iff $\forall q, r \in \mathcal{P}$, $q \sim r$ and $\bigcup \mathcal{P} \sim p$

We do not assume that $\succsim$ is monotonic with respect to set inclusion, nor that $\varnothing$ is minimal and Ω maximal. Of course, the axiomatisation to follow will be consistent with these assumptions, but it's not needed to get the Ramseyan proposal off the ground. And one can certainly *imagine* an agent who is, e.g., less than certain of Ω.

We do, however, assume that if a proposition is maximal then α is certain of its truth. Given this, we can now restate Ramsey's idea: p is believed to degree $^n/_m$ iff

$$p \sim q_1 \cup \cdots \cup q_n,$$

where the $q_1, \ldots, q_n$ belong to an m-scale $\{q_1, \ldots, q_n, \ldots, q_m\}$ of some maximal proposition q. A good start—but there's a natural extension that will be helpful to incorporate into what follows. Consider this case:

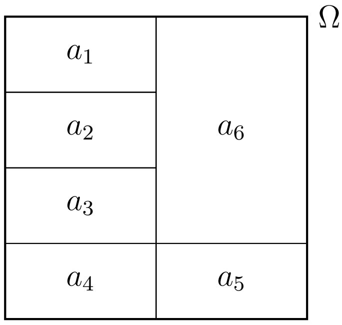


We have a simple algebra of sets with three atoms, a_1, a_2, and a_3, and

$$\Omega \succ a_1 \cup a_3 \sim a_2 \cup a_3 \succ a_3 \sim a_1 \cup a_2 \succ a_1 \sim a_2 \succ \varnothing$$

Assuming reflexivity, $\{\Omega\}$ is a 1-scale of Ω, and $\{a_3, a_1 \cup a_2\}$ is a 2-scale of Ω, so Ramsey would have us say that $\mathcal{C}r(\Omega) = 1$ and $\mathcal{C}r(a_3) = \mathcal{C}r(a_1 \cup a_2) = {}^1\!/_2$. a_1 and a_2 do not belong to any n-scale of Ω, so we do not yet have any purchase on the strength with which they're believed. However, $\{a_1, a_2\}$ is a 2-scale of $a_1 \cup a_2$, so it's only reasonable to say that $\mathcal{C}r(a_1) = \mathcal{C}r(a_2) = {}^1\!/_4$.

Or consider the following case:



Here, assume Ω is maximal and $\varnothing$ minimal, and $\succsim$ includes:

$$a_5 \cup a_6 \sim a_1 \cup a_2 \cup a_3 \cup a_4 \succ a_6 \sim a_1 \cup a_2 \cup a_3 \succ a_1 \sim a_2 \sim a_3 \sim a_4 \sim a_5$$

$\{a_5 \cup a_6, a_1 \cup a_2 \cup a_3 \cup a_4\}$ is a 2-scale of Ω, and a_1, a_2, a_3 are 3 members of the 4-scale $\{a_1, a_2, a_3, a_4\}$ of $a_1 \cup a_2 \cup a_3 \cup a_4$; we would therefore like to say that $\mathcal{C}r(a_1 \cup a_2 \cup a_3) = {}^3\!/_8$. We note that $\{a_6\}$ is a 1-scale of $a_1 \cup a_2 \cup a_3$; hence, $\mathcal{C}r(a_6) = {}^3\!/_8$.

We can capture the foregoing points by means of the following definition:[10]

Definition 5.2 For integers n, m such that $m \geq n \geq 0$, $m > 0$, p is
 (i) ${}^0\!/_m$*-valued* if p is minimal and ${}^m\!/_m$*-valued* if p is maximal
 (ii) ${}^n\!/_m$*-valued* if $p \sim q_1 \cup \cdots \cup q_{n'}$, where the $q_1, \ldots, q_{n'}$ belong to an m'-scale of an ${}^{n''}\!/_{m''}$-valued proposition, and ${}^{(n' \cdot n'')}\!/_{(m' \cdot m'')} = {}^n\!/_m$

The generalised version of Ramsey's suggestion now amounts to:

$$\alpha \text{ believes } p \text{ to degree } {}^n\!/_m \text{ if } p \text{ is } {}^n\!/_m\text{-valued}$$

As such, define a *Ramsey function* as follows:

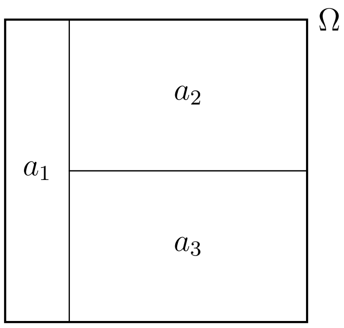
Definition 5.3 $\mathcal{C}r : \boldsymbol{\mathcal{B}} \mapsto [0, 1]$ is a *Ramsey function* (relative to $\succsim$) iff, for all $p \in \boldsymbol{\mathcal{B}}$, if p is ${}^n\!/_m$-valued, then $\mathcal{C}r(p) = {}^n\!/_m$

[10] With a rich enough preference structure, we could also accommodate irrationally-valued propositions. For an irrational real r, say that p is r-valued just in case (i) $p \prec q$ for every ${}^n\!/_m$-valued q such that ${}^n\!/_m > r$, and (ii) $p \succ q$ for every ${}^n\!/_m$-valued q such that $r < {}^n\!/_m$. But this definition is plausible only under the supposition that there is some ${}^n\!/_m$-valued q for every rational fraction ${}^n\!/_m$ (such that $m \geq n \geq 0$). Thanks are due to Nicholas DiBella here; see (DiBella MS) for additional discussion.

The connection to the GRP is immediate. In fact, in the terminology of Definition 5.1, the first (non-inductive) clause of the GRP states that for $m \geq n$, p is believed ${}^{n}\!/\!{}_{m}$ times as much as q whenever $\mathfrak{P}$ is an m-scale of q, and $\mathfrak{P}' \subseteq \mathfrak{P}$ is an n-scale of p. In this case, Definition 5.2 says that if q is ${}^{n'}\!/\!{}_{m'}$-valued, then p is ${}^{(n \cdot n')}\!/\!{}_{(m \cdot m')}$-valued—that is, for any Ramsey function $\mathcal{C}r$,

$$\mathcal{C}r(p) = {}^{n}\!/\!{}_{m} \cdot \mathcal{C}r(q)$$

Any Ramsey function recaptures the GRP *almost* in its entirety, with one limitation: it cannot account for quantitative comparisons between propositions that are not ${}^{n}\!/\!{}_{m}$-valued, for some n, m. Ultimately, a Ramsey function scales every (non-minimal) ${}^{n}\!/\!{}_{m}$-valued proposition relative to the (or one of the) maximal proposition(s). With respect to pairs of propositions that cannot be so scaled, it's possible for a Ramsey function to fail to cohere with the GRP. An example of this would be the following:



Where in this case,

$$\Omega \succ a_2 \cup a_3 \succ a_1 \cup a_2 \sim a_1 \cup a_3 \succ a_2 \sim a_3 \succ a_1 \succ \varnothing$$

The only non-trivial n-scale here is the 2-scale $\{a_2, a_3\}$ of $a_2 \cup a_3$; but, since $a_2 \cup a_3$ cannot be assigned a definite value relative to Ω, the values of a_2, a_3 and $a_2 \cup a_3$ are likewise left indeterminate.[11]

Ramsey doesn't say anything about those circumstances where p is not ${}^{n}\!/\!{}_{m}$-valued, and this is a lacuna in the present proposal—though perhaps not a very troubling one. One might assume that such cases don't exist. Let $\mathfrak{B}^\star \subseteq \mathfrak{B}$ designate the set of ${}^{n}\!/\!{}_{m}$-valued propositions. Then, the assumption would be:

B1. $\mathfrak{B}^\star = \mathfrak{B}$

B1 is not implied by A1-A5, and in that sense is a stronger condition than required for probabilistic coherence. However, it's worth noting that B1 *is* implied by the following continuity assumption, used in Stefánsson's (2017, forthcoming) defence of probabilistic comparativism:[12]

[11] Below we'll add an axiom (B3) to ensure that $\mathcal{C}r(a_2) = \mathcal{C}r(a_3)$, but not enough to guarantee that $\mathcal{C}r(a_2 \cup a_3) = 2 \cdot \mathcal{C}r(a_2)$. Note that in this kind of case, there will be more than one probability function that agrees with $\succsim$; each is a Ramsey function relative to $\succsim$.

[12] The proof of this is straightforward: let q be maximal, and consider every pair $p, q \in \mathfrak{B}$ where p is non-minimal; Continuity then states that $\{p\}$ is a 1-scale of the union of n members of an m-scale of q. Continuity is explicitly assumed in (Stefánsson forthcoming) under the title 'Savage Continuity', and also implied (in the context of A1-A5) by 'Suppes Continuity'.

Continuity: For all non-minimal $p, q \in \mathcal{B}$, there are $p', q' \in \mathcal{B}$ such that $p \sim p', q \sim q'$, and p' and q' are each the union of some subset of a finite set of disjoint propositions $\{r_1, \dots, r_n\}$ such that $r_i \sim r_j$ for $i, j = 1, \dots, n$

I suspect that B1 is close to correct in many ordinary cases, and we can treat it as a reasonable idealisation for now. Below, I'll show how to do without it. Nevertheless, B1 only ensures that every $p \in \mathcal{B}$ is n/m-valued; it isn't yet enough to ground a plausible comparativist story. There are two problems that can arise in the absence of any additional assumptions about $\succsim$.

First: nothing has been said to guarantee that Definition 5.3 is *consistent*. For note that, without further assumptions, it's entirely possible for, e.g., $p \sim q$, where for some r, p belongs to a 2-scale of r and q belongs to a 3-scale of r. This is clearly undesirable: α can't believe p to the precise degrees $1/2$ and $1/3$ simultaneously! If Ramsey functions are to be well defined, we'll need to ensure that if p is n/m-valued and n'/m'-valued, then $n/m = n'/m'$.

Second: nothing has been said to guarantee that a Ramsey function relative to $\succsim$ will *agree* with $\succsim$. Indeed, nothing ensures that $\mathcal{C}r(p) \geq \mathcal{C}r(q)$ if *or* only if $p \succsim q$. For example, p could be $1/2$-valued, and q $1/4$-valued, yet $q \succsim p$. This is wholly unacceptable: if the order of the numerical values we assign to partial beliefs doesn't at least match up to the belief ranking, then there's no natural sense in which those values represent the *strengths* with which those propositions are believed.

In the context of B1, we can kill these two birds with one stone, using the following (quite strong) axiom:

B2. If p is n/m-valued and q is n'/m'-valued, then $p \succsim q$ iff $n/m \geq n'/m'$

B2 implies that $\succsim$ is transitive and complete over $\mathcal{B}^\star$, and, interestingly, that if p and q belong to an n-scale of r and r is non-minimal, then $p \not\subseteq q$.[13] There are as such some logical restrictions on what kinds of propositions we can 'add' using the Ramseyan process—it's not the case that "anything goes". More importantly, B2 is obviously necessary (and given B1, sufficient) to avoid the two foregoing problems, as the following theorem (proven in Appendix A) shows:

Theorem 5.1 *If $\succsim$ is a binary relation on $\mathcal{B} \subseteq \wp(\Omega)$, then $\succsim$ satisfies B2 iff there exists a function $\mathcal{C}r : \mathcal{B} \mapsto \mathbb{R}$ such that:*

(i) *$\mathcal{C}r$ is a Ramsey function with respect to $\succsim$, and*

(ii) *For all $p, q \in \mathcal{B}^\star$, $p \succsim q$ iff $\mathcal{C}r(p) \geq \mathcal{C}r(q)$*

Furthermore, $\mathcal{C}r$ is the unique Ramsey function relative to $\succsim$ that agrees with $\succsim$ iff $\succsim$ satisfies B1

It is easy to see that B2 is implied already by the axioms A1-A5 (likewise A2-A5*). To see that B1 and B2 are consistent with non-probabilistic Ramsey

[13] B2 entails that if an n-scale $\mathcal{P}$ of r contains minimal propositions, then r and every $p \in \mathcal{P}$ is minimal. If p, q belong to some n-scale $\mathcal{P}$ of r, then if $p \subseteq q$, $p \cap q = p$. Since p is minimal, r is minimal.

functions, consider the following simple example. Let $\mathcal{C}r$'s domain be as follows:[14]

$$\mathfrak{B} = \{\varnothing, p, q, p \cap q, p \cup q, \Omega\}$$

Now suppose that $\succsim$ is transitive and reflexive, and:

$$p \cup q \succ \Omega \sim p \sim q \succ p \cap q \sim \varnothing$$

$\succsim$ satisfies B1 and B2. Since $\{p, q\}$ is a 2-scale of the maximal proposition $p \cup q$, $\mathcal{C}r(p \cup q) = 1$ and $\mathcal{C}r(p) = \mathcal{C}r(q) = 1/2$. Ω is neither maximal nor a member of any non-trivial n-scale, but it's just as likely as p; hence $\mathcal{C}r(\Omega) = 1/2$.

Essentially, B2 imposes a limited kind of qualitative additivity on $\succsim$, specifically with respect to relations between propositions constructed out of the same n-scale of an n'/m'-valued proposition. Roughly: *within* an n-scale, $\succsim$ behaves 'probabilistically'—but not every proposition can be constructed out of members of an appropriate n-scale, and across n-scales $\succsim$ can behave quite irrationally.

On the basis of Theorem 5.1, we could characterise a comparativist view as *Ramseyan* whenever it implies:

> **Ramseyan Comparativism:** If α's belief ranking $\succsim$ satisfies B1 and B2 and $\mathcal{C}r$ is a Ramsey function relative to $\succsim$, then $\mathcal{C}r$ is an adequate model of α's beliefs *simpliciter*

But I think we can do better still than Ramseyan comparativism, and adopt a set-of-functions representation of $\succsim$ for the cases where B1 fails. For this, we will need the following axiom:

> **B3.** $\succsim$ is a preorder

B3 is obviously necessary if *any* real-valued function or set thereof is to agree with $\succsim$, regardless of whatever other restrictions we want to place on that relation. For simplicity, we focus on the case where $\mathfrak{B}$ is countable; thus,

> **Theorem 5.2** *If $\succsim$ is a binary relation on a countable set $\mathfrak{B} \subseteq \wp(\Omega)$, then $\succsim$ satisfies B2 and B3 iff there exists a non-empty set $\mathfrak{F}$ of functions into $[0, 1]$ that agrees with $\succsim$, where every $\mathcal{C}r \in \mathfrak{F}$ is a Ramsey function relative to $\succsim$*

A proof is provided in Appendix B. Note of course that whenever B2 and B3 are satisfied, there will be a unique $\mathfrak{F}$ that's maximal with respect to inclusion. So, we let *imprecise-Ramseyan comparativism* denote any comparativist view that implies:

[14] For the purposes of Theorem 5.1, $\mathfrak{B}$ can be any subset of $\wp(\Omega)$. For the present example, supposing that $\mathfrak{B}$ is an algebra of sets would not change the point (e.g., we could set every other proposition equal to $\varnothing$).

Imprecise-Ramseyan Comparativism: If a non-empty set $\mathfrak{F}$ of functions into $[0,1]$ agrees with α's belief ranking $\succsim$, where every $\mathcal{C}r \in \mathfrak{F}$ is a Ramsey function relative to $\succsim$, and $\mathfrak{F}$ is maximal with respect to inclusion, then $\mathfrak{F}$ is an adequate model of α's beliefs *simpliciter*

According to imprecise-Ramseyan comparativism, any proposition $p \notin \mathfrak{B}^\star$ will not usually be assigned a precise numerical value, though we can still supervaluate over $\mathfrak{F}$ to generate 'imprecise' strengths of belief.

Furthermore, if any probability function $\mathcal{C}r$ almost agrees with $\succsim$, then $\succsim$ satisfies B2 and B3, and $\mathcal{C}r$ is *ipso facto* a Ramsey function relative to $\succsim$. This is a nice result to have: a Ramsey function representation of $\succsim$ never *conflicts* with a probability function, or set of probability functions, with respect to $^n/_m$-valued propositions. Moreover, most (precise and imprecise) probabilistic comparativists will at least want to say that if a probability function $\mathcal{C}r$ agrees with α's continuous belief ranking, then $\mathcal{C}r$ adequately represents α's beliefs. The (imprecise) Ramseyan comparativist can say exactly this, *without* supposing that $\succsim$ satisfy conditions that are as strong as A1-A5 or A2-A5*.

The cost, of course, is that the Ramseyan comparativist—without further additions to the position as outlined here—has to give up on quantitative comparisons between pairs of propositions that are not $^n/_m$-valued. B2 and B3 are not sufficient for *total* coherence with the GRP if any such propositions exist, but they *are* necessary if we make some very minimal scaling assumptions:

Theorem 5.3 *If $\mathcal{C}r$ coheres with the GRP, then at least one of the following is false:*

(i) *$\succsim$ violates B2*
(ii) *There are $p, q \in \mathfrak{B}$ such that $p \succ q$*
(iii) *If p is minimal, then $\mathcal{C}r(p) = 0$*
(iv) *$\mathcal{C}r$ agrees with $\succsim$*

Furthermore, if a set of real-valued functions $\mathfrak{F}$ coheres with the GRP, then at least one of (i), (ii), (v), *or* (vi) *is false, where:*

(v) *If p is minimal, then $\forall \mathcal{C}r \in \mathfrak{F}$, $\mathcal{C}r(p) = 0$*
(vi) *$\mathfrak{F}$ agrees with $\succsim$*

6 Conclusion

A belief ranking that satisfies B2 and B3 has only a very weak 'additive' structure, and it's unclear how we could remove even these restrictions while preserving enough structure with respect to the union of epistemically exclusive sets to justify treating it as even a *limited* qualitative analogue of addition. If this is right, then we have an initial answer to the question posed in §4: the analogy with addition thoroughly breaks down when either B2 or B3 are violated.

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Appendix A: Theorem 5.1

Existence, left-to-right: Assume B2. If p is ${}^{n}/_{m}$-valued and ${}^{n'}/_{m'}$-valued, then ${}^{n}/_{m} = {}^{n'}/_{m'}$. So it's possible to assign a unique $r \in \mathbb{R}$ to every $p \in \mathfrak{B}^{\star}$ so as to define a Ramsey function $\mathcal{C}r$ relative to the restriction of $\succsim$ to $\mathfrak{B}^{\star}$. $\mathcal{C}r$ can then be extended from $\mathfrak{B}^{\star}$ to $\mathfrak{B}$ consistent with that function being a Ramsey function relative to the entirely of $\succsim$ (e.g., let $\mathcal{C}r(p) = 0$ for all $p \notin \mathfrak{B}^{\star}$). This establishes clause (i). Now suppose that for $p, q \in \mathfrak{B}^{\star}$, $p \succsim q$. Since for some n, m, n', m', p is ${}^{n}/_{m}$-valued and q is ${}^{n'}/_{m'}$-valued, so ${}^{n}/_{m} \geq {}^{n'}/_{m'}$. By (i), $\mathcal{C}r(p) \geq \mathcal{C}r(q)$. Next suppose that $\mathcal{C}r(p) \geq \mathcal{C}r(q)$. Since $\mathcal{C}r$ is a Ramsey function, p is ${}^{n}/_{m}$-valued and q is ${}^{n'}/_{m'}$-valued, for ${}^{n}/_{m} \geq {}^{n'}/_{m'}$. So, from B2, $p \succsim q$. This establishes clause (ii).

Existence, right-to-left: Suppose $\mathcal{C}r : \mathfrak{B} \mapsto \mathbb{R}$ satisfies (i) and (ii). Next suppose that p is ${}^{n}/_{m}$-valued and q is ${}^{n'}/_{m'}$-valued. So, $\mathcal{C}r(p) = {}^{n}/_{m}$ and $\mathcal{C}r(q) = {}^{n'}/_{m'}$. Since $\mathcal{C}r$ agrees with $\succsim$ over $\mathfrak{B}^{\star}$, so ${}^{n}/_{m} \geq {}^{n'}/_{m'}$ iff $p \succsim q$.

Uniqueness: The left-to-right is obvious by consideration of its contrapositive and Definition 5.3. The restriction of $\mathcal{C}r$ to $\mathfrak{B}^{\star}$ is the unique Ramsey function relative to the restriction of $\succsim$ to $\mathfrak{B}^{\star}$; so if $\mathfrak{B}^{\star} = \mathfrak{B}$ then $\mathcal{C}r$ is the unique Ramsey function that agrees with $\succsim$ *simpliciter*. □

Appendix B: Theorem 5.2

Existence, left-to-right: Assume $\succsim$ satisfies B2, B3, and $\mathfrak{B}$ is countable. If B1, then the existence of the set $\mathfrak{F}$ follows from the uniqueness condition of Theorem 5.1. We therefore focus on the case where $\mathfrak{B}^{\star} \subset \mathfrak{B}$. From B3, there's at least one non-empty set $\mathfrak{G}$ of functions $\mathcal{C}r : \mathfrak{B} \mapsto \mathbb{R}$ that agrees with $\succsim$. A proof of this can be found in (Dubra et al. 2004, p.556). What we need to show is that there exists a non-empty subset $\mathfrak{G}^{*}$ of $\mathfrak{G}$ such that:

1. $\mathfrak{G}^{*}$ agrees with $\succsim$
2. $\forall \mathcal{C}r \in \mathfrak{G}^{*}$, there's a strictly increasing transformation $\mathcal{C}r'$ of $\mathcal{C}r$ s.t.:
   (a) $\mathcal{C}r'$ is bounded above by 1 and below by 0
   (b) $\mathcal{C}r'$ is a Ramsey function with respect to $\succsim$

The set $\mathfrak{F}$ of all such transformations will agree with $\succsim$, completing the proof.

There are three cases to consider:

1. $\mathfrak{B}^{\star}$ is empty
2. $\mathfrak{B}^{\star}$ contains only minimal and/or maximal elements of $\mathfrak{B}$
3. $\mathfrak{B}^{\star}$ contains non-minimal, non-maximal elements of $\mathfrak{B}$

The first two are straightforward and omitted. For the third case, note that if $\mathfrak{G}$ agrees with $\succsim$ and $p \succ q$, then:

(i) $\mathcal{C}r(p) \geq \mathcal{C}r(q)$, for all $\mathcal{C}r \in \mathfrak{G}$
(ii) $\mathcal{C}r(p) > \mathcal{C}r(q)$, for at least one $\mathcal{C}r \in \mathfrak{G}$

Hence, for any $\mathcal{C}r \in \mathfrak{G}$, if $p \succ q$ then either $\mathcal{C}r(p) > \mathcal{C}r(q)$ or $\mathcal{C}r(p) = \mathcal{C}r(q)$. For $p, q \in \mathfrak{B}^\star$, B2 implies that for any Ramsey function, if $p \succ q$, then $\mathcal{C}r(p) > \mathcal{C}r(q)$; so, it's not in general true that if $\mathfrak{G}$ agrees with $\succsim$, then for every $\mathcal{C}r \in \mathfrak{G}$ there will be a strictly increasing transformation of $\mathcal{C}r$ that's also a Ramsey function with respect to $\succsim$. But define $\mathfrak{G}^* \subseteq \mathfrak{G}$ as follows:

$$\mathfrak{G}^\star = \{\mathcal{C}r \in \mathfrak{G} : \text{if } p, q \in \mathfrak{B}^\star \text{ and } p \succ q, \text{ then } \mathcal{C}r(p) > \mathcal{C}r(q)\}$$

$\mathfrak{G}^\star$ agrees with $\succsim$, and by (ii), we know that it's non-empty. If we let $\mathfrak{G}^\circ$ denote the set of restrictions of every $\mathcal{C}r \in \mathfrak{G}^\star$ to $\mathfrak{B}^\star$, then the unique Ramsey function $\mathcal{C}r^\star$ on $\mathfrak{B}^\star$ is a strictly increasing transformation of every $\mathcal{C}r \in \mathfrak{G}^\circ$. So we just have to show that for each $\mathcal{C}r \in \mathfrak{G}^\star$, there's an extension of $\mathcal{C}r^\star$ from $\mathfrak{B}^\star$ to $\mathfrak{B}$ that's a strictly increasing transformation of $\mathcal{C}r$. Let $\mathcal{C}r$ be any function in $\mathfrak{G}^\star$. For any set of non-minimal, non-maximal propositions $\mathfrak{P} = \{p_1, p_2, \dots\} \subseteq \mathfrak{B}$, there's a unique pair $q, r \in \mathfrak{B}^\star$ such that:

(i) $q \succ p_i \succ r$, for all $p_i \in \mathfrak{P}$
(ii) There's no $s \in \mathfrak{B}^\star$ such that $q \succ s \succ p_i$ or $p_i \succ s \succ q$, for all $p_i \in \mathfrak{P}$

So for any $p_i \in \mathfrak{P}$, $\mathcal{C}r(q) > \mathcal{C}r(r)$ and $\mathcal{C}r(q) \geq \mathcal{C}r(p_i) \geq \mathcal{C}r(r)$. From the fact that $\mathcal{C}r^\star(q), \mathcal{C}r^\star(r)$ are rational and $\mathcal{C}r^\star(q) \neq \mathcal{C}r^\star(r)$, for any way the $\mathcal{C}r(p_i)$ might be ordered between $\mathcal{C}r(q)$ and $\mathcal{C}r(r)$, there are sufficient real values between $\mathcal{C}r^\star(q)$ and $\mathcal{C}r^\star(r)$ to recreate that order.

Existence, right-to-left: Suppose there's a non-empty set $\mathfrak{F}$ of functions into $[0, 1]$ that agrees with $\succsim$, where every $\mathcal{C}r \in \mathfrak{F}$ is a Ramsey function relative to $\succsim$. That $\succsim$ satisfies B3 is straightforward. Where $\mathfrak{B}^\star$ is empty, B2 is trivially satisfied. So, suppose $\mathfrak{B}^\star \neq \varnothing$. For every $\mathcal{C}r \in \mathfrak{F}$, $\mathcal{C}r$ is a Ramsey function with respect to $\succsim$, so if p is ${}^{n}/_{m}$-valued and q ${}^{n'}/_{m'}$-valued, $\mathcal{C}r(p) = {}^{n}/_{m}$ and $\mathcal{C}r(p) = {}^{n'}/_{m'}$. Since $\mathfrak{F}$ agrees with $\succsim$, $p \succsim q$ iff ${}^{n}/_{m} \geq {}^{n'}/_{m'}$. □

Appendix C: Theorem 5.3

For the first part of the theorem, we prove the contrapositive. Suppose first that $\succsim$ violates B2, and that $\mathcal{C}r$ agrees with $\succsim$. From the falsity of B2, there exists a pair p, q such that p is ${}^{n}/_{m}$-valued, q is ${}^{n'}/_{m'}$-valued, and $p \succsim q \not\leftrightarrow {}^{n}/_{m} \geq {}^{n'}/_{m'}$. There are three cases to consider:

(1) Neither p nor q is minimal
(2) Both p and q are minimal
(3) Exactly one of p or q is minimal

Start with (1). Focus first on p, and let r henceforth designate some maximal proposition. (If p is ${}^{n}/_{m}$-valued and non-minimal, then a maximal proposition exists.) Since it's ${}^{n}/_{m}$-valued, p is either:

(i) The union of n members of an m-scale of r, or
(ii) The union of n'' members of an m''-scale of ... the union of n''' members of an m'''-scale of r

In case (i), $\mathcal{C}r$ coheres with the GRP only if $\mathcal{C}r(p) = {}^{n}/_{m} \cdot \mathcal{C}r(r)$; in case (ii), only if $\mathcal{C}r(p) = {}^{(n'' \cdot \ldots \cdot n''')}/_{(m'' \cdot \ldots \cdot m''')} \cdot \mathcal{C}r(r)$, where ${}^{(n' \cdot \ldots \cdot n'')}/_{(m' \cdot \ldots \cdot m'')} = {}^{n}/_{m}$. The same reasoning applies to q, *mutatis mutandis*, so $\mathcal{C}r$ coheres with the GRP only if $\mathcal{C}r(p) = {}^{n'}/_{m'} \cdot \mathcal{C}r(r)$. Assume for the sake of *reductio* that $\mathcal{C}r$ coheres with the GRP. Now suppose ${}^{n}/_{m} \geq {}^{n'}/_{m'}$, so $\mathcal{C}r(p) \geq \mathcal{C}r(q)$, and (since $\mathcal{C}r$ agrees with $\succsim$) so $p \succsim q$. In the other direction, suppose $p \succsim q$; so $\mathcal{C}r(p) \geq \mathcal{C}r(q)$, and ${}^{n}/_{m} \geq {}^{n'}/_{m'}$. So, $p \succsim q \leftrightarrow {}^{n}/_{m} \geq {}^{n'}/_{m'}$, which violates our assumptions above. So $\mathcal{C}r$ does not cohere with the GRP.

Now case (2). Add now the assumptions that there are $p, q \in \mathfrak{B}$ such that $p \succ q$, and that if p is minimal, then $\mathcal{C}r(p) = 0$. If p and q are both minimal then $p \sim q$, and if $\mathcal{C}r$ agrees with $\succsim$ then $\mathcal{C}r(p) = \mathcal{C}r(q) > \mathcal{C}r(s)$, for any s such that $s \not\prec p$ (and hence $s \succ p$). Since p and q are both ${}^{0}/_{m}$-valued by definition, the only way B2 might be violated is if at least one of the two propositions is also ${}^{n}/_{m}$-valued, for some $n > 0$. Suppose this is the case of p; then by the earlier reasoning, $\mathcal{C}r$ coheres with the GRP only if $\mathcal{C}r(p) = {}^{n}/_{m} \cdot \mathcal{C}r(r)$. Since ${}^{n}/_{m} > 0$ and $\mathcal{C}r(r) > 0$, this is false; so $\mathcal{C}r$ does not cohere with the GRP.

Case (3) is then straightforward given the above. Likewise, the second part of the theorem (for sets of functions) follows the same structure as the proof just given with only minor changes, and can be omitted.