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1 Partial Belief and Subjective Probability

In the posthumously published 'Truth and Probability' (1926), Ramsey sets out an influential account of the nature, measurement, and norms of partial belief. The essay is a foundational work on subjectivist interpretations of probability, according to which probabilities can be interpreted as rational degrees of belief (see entry on Interpretations of Probability). Many of its key ideas and arguments have since featured in other foundational works within the subjectivist tradition (e.g., Savage 1954, Jeffrey 1965).

Ramsey's central claim in 'Truth and Probability' is that the laws of probability supply us with a 'logic of partial belief'. That is, the laws specify what would need to be true of any consistent set of partial beliefs, in a manner analogous to how the laws of classical logic might be taken to generate necessary conditions on any consistent set of full beliefs. His case for this is based on a novel account of what partial beliefs are and how they can be measured.

The guiding idea of Ramsey's view is that 'the degree of a belief is a causal property of it, which we can express vaguely as the extent to which we are prepared to act on it' (p. 169). However, a vague expression of the idea will not suffice, and Ramsey argues that 'the degree of a belief ... has no precise meaning unless we specify more exactly how it is to be measured' (p. 167). To precisify his account, therefore, Ramsey sketches an original and influential procedure for the measurement of partial beliefs. The procedure takes as input the subject's ordinal preferences over (at least) (i) propositions that are maximally specific with respect to matters the subject cares about (henceforth referred to as *worlds*), (ii) *binary gambles* of the form 'world ω_1 if p, and world ω_2 if $\neg p$', and (iii) *trinary gambles* of the form 'ω_1 if $p \,\&\, q$, ω_2 if $p \,\&\, \neg q$, and ω_3 if $\neg p$', and outputs precise numerical representations of her partial beliefs and utilities. It can be roughly summarised as follows:

1. Determine the subject's ordinal preferences by offering a sequence choices between pairs of worlds and/or gambles.

2. Assuming the subject is an expected utility maximiser, use her preferences to define numerical utilities for each world and gamble.

3. Define (i) the subject's degree of belief towards p in terms of her utilities for binary gambles involving p, and (i) her degree of belief towards p given q using her utilities for binary and trinary gambles involving p and q.

The following paragraphs will discuss these steps in some more detail; thorough treatments of Ramsey's procedure can be found in (Sahlin 1990), (Bradley 2001), and (Elliott 2017).

Regarding the first step, Ramsey says:

> [Suppose] that our subject has certain beliefs about everything; then he will act so that what he believes to be the total consequences of his action will be the best possible. If then we had the power of the Almighty, and could persuade our subject of our power, we could, by offering him options, discover how he placed in order of merit all possible courses of the world. In this way all possible worlds would be put in an order of value... Suppose next that the subject is capable of doubt; then we could test his degree of belief in different propositions by making him offers of the following kind. Would you rather have world [ω_1] in any event; or world [ω_2] if p is true, and world [ω_3] if p is false? (p. 177)

A common objection to this early stage of the procedure is that if the experimenter were to convince their subject they 'had the power of the Almighty', then this would substantial cause changes to the beliefs being measured (e.g., Jeffrey 1983, pp. 158-60; cf. Sobel 1998, p. 255-6; Rabinowicz & Eriksson 2013). For instance, the process would certainly alter the subject's beliefs regarding 'There are omnipotent beings, and one of them is before me', with potentially many ripple on effects through to her other beliefs. Ramsey makes note of a similar problem for an alternative gambling-based measurement procedure, but for unknown reasons does not discuss the problem as it arises for his own account.

Regarding the second stage, Ramsey's procedure requires a suite of assumptions about the partial beliefs, preferences, and utilities of the subject. In his words,

> I propose to take as a basis a general psychological theory, which is now universally discarded, but nevertheless comes, I think, fairly close to the truth in the sort of cases with which we are most concerned. I mean the theory that we act in the way we think most likely to realize the objects of our desires, so that a person's actions are completely determined by his desires and opinions. (p. 174)

This greatly underspecifies the 'general psychological theory' that Ramsey implicitly relies on to justify his later definitions, which can be put more precisely as:

A1. There exists a real-valued *utility function* $\mathcal{U}$, defined for gambles and worlds, that represents the subject's utilities on an interval scale[1] and

[1] $\mathcal{U}$ represents utilities on an interval scale whenever $\mathcal{U}(p) - \mathcal{U}(q) \geq \mathcal{U}(r) - \mathcal{U}(s)$ iff the difference in utility between p and q is at least as great as the difference between r and s.

satisfies the condition:

(i) If p implies q, then $\mathcal{U}(p) = \mathcal{U}(p \,\&\, q)$

A2. There exists a real-valued *subjective probability function* $\mathcal{P}r$ that represents the subject's partial beliefs, and satisfies the conditions:

(i) $\mathcal{P}r(p) = 1 - \mathcal{P}r(\neg p)$

(ii) $\mathcal{P}r(p \text{ given } q) = \mathcal{P}r(p \,\&\, q)/\mathcal{P}r(q)$, whenever $\mathcal{P}r(p) > 0$

(iii) $\mathcal{P}r(p \text{ given } q) = 1 - \mathcal{P}r(\neg p \text{ given } q)$

A4. The subject *maximises expected utility*, in the specific sense that the utility of a gamble 'ω_1 if p, ω_2 if $\neg p$' is equal to

$$\mathcal{U}(\omega_1 \,\&\, p)\mathcal{P}r(p) + \mathcal{U}(\omega_2 \,\&\, \neg p)\mathcal{P}r(\neg p),$$

and likewise for trinary gambles, *mutatis mutandis.*

One of Ramsey's most important insights is that, under these assumptions, the subject's preferences will satisfy a small set of relatively simple axioms only if her utilities and partial beliefs are a certain way—thus, it is in principle possible to use information about her preferences to determine facts about her partial beliefs. While the assumptions clearly involve some idealisations, Ramsey justifies these by noting that the development of any measurement process 'cannot be accomplished without introducing a certain amount of hypothesis or fiction' (p. 168), and that the assumptions come close enough to the truth to render them still useful (p. 173).

The reasoning by which we go from preferences to a numerical representation of utilities is complicated, and in Ramsey's paper mostly left unstated. In brief, it begins with the notion of an *ethically neutral proposition*, which can here be defined as:

Ethical Neutrality. p is *ethically neutral* for a subject iff, for all worlds ω consistent with p and $\neg p$, the subject is indifferent between ω, $\omega \,\&\, p$, and $\omega \,\&\, \neg p$.

Where p is ethically neutral and the subject not indifferent between ω_1 and ω_2, we can show $\mathcal{P}r(p) = \frac{1}{2}$ whenever the subject is indifferent between 'ω_1 if p, ω_2 if $\neg p$' and 'ω_2 if p, ω_1 if $\neg p$'. To see this, note that under assumptions A1 and A3, the stated indifference holds only if and only if

$$\mathcal{U}(\omega_1 \,\&\, p)\mathcal{P}r(p) + \mathcal{U}(\omega_2 \,\&\, \neg p)\mathcal{P}r(\neg p) \;=\; \mathcal{U}(\omega_2 \,\&\, p)\mathcal{P}r(p) + \mathcal{U}(\omega_1 \,\&\, \neg p)\mathcal{P}r(\neg p).$$

Given then A1 and A2, and assuming p is ethically neutral, this reduces to

$$\mathcal{U}(\omega_1)\mathcal{P}r(p) + \mathcal{U}(\omega_2)(1 - \mathcal{P}r(p)) \;=\; \mathcal{U}(\omega_2)\mathcal{P}r(p) + \mathcal{U}(\omega_1)(1 - \mathcal{P}r(p)),$$

where $\mathcal{U}(\omega_1) \neq \mathcal{U}(\omega_2)$. Algebra then dictates that $\mathcal{P}r(p) = \frac{1}{2}$.

Having characterised a class of ethically neutral propositions of probability $\frac{1}{2}$, Ramsey is able to define in terms of preferences when the difference in utility between two worlds ω_1 and ω_2 is equal to that between ω_3 and ω_4. This provides him with the resources needed to sketch a representation theorem, according to which the subject's preferences satisfy eight axioms only if there exists a utility function $\mathcal{U}$ such that the difference in utility between ω_1 and ω_2 is at least as great as that between ω_3 and ω_4 if and only if

$$\mathcal{U}(\omega_1) - \mathcal{U}(\omega_2) \geq \mathcal{U}(\omega_3) - \mathcal{U}(\omega_4)$$

Included amongst Ramsey's axioms are obvious conditions, such as that preferences ought to be transitive; as well as less obvious conditions, e.g., that for every pair of worlds ω_1, ω_2, there's a ω_3 whose utility is exactly halfway between that of ω_1 and ω_2.

There has been very little empirical investigation into Ramsey's axioms, primarily due to the widespread opinion that his results have been superseded by those of Savage (1954) and later decision theorists (see, e.g., Fishburn 1981). However, Ramsey's assumption that there exists an ethically neutral proposition of probability $\frac{1}{2}$ (needed for the definition of equal differences in utility to make sense) has attracted substantial critical discussion. Ramsey provides his readers with no reasons to believe that even one ethically neutral proposition exists, still less that their existence is a precondition for the *consistency* of partial beliefs. (For discussion, see Sobel 1998; Bradley 2001; Eriksson & Hájek 2007; Elliott 2017.) Ethically neutral propositions have played an equally central role in several representation theorems developed since Ramsey's, including those of Davidson, Suppes & Siegal (1957), Debreu (1959), and Fishburn (1967).

The third and final stage of Ramsey's procedure takes us from the utility function $\mathcal{U}$ to a definition of the subject's degrees of belief:

> Having thus defined a way of measuring value we can now derive a way of measuring belief in general. If the option of [ω_2] for certain is indifferent with that of ['ω_1 if p, ω_3 if $\neg p$'], we can define the subject's degree of belief in p as the ratio of the difference between [ω_2] and [ω_3] to that between [ω_1] and [ω_3]. (p. 179)

In a footnote, Ramsey adds that ω_1 must imply p, and ω_3 must imply $\neg p$. The definition again makes sense in light of assumptions A1-A3 above: ω_2 has the same utility as 'ω_1 if p, ω_3 if $\neg p$' if and only if

$$\mathcal{U}(\omega_2) = \mathcal{U}(\omega_1)\mathcal{P}r(p) + \mathcal{U}(\omega_3)(1 - \mathcal{P}r(p))$$

Where $\mathcal{U}(\omega_1) \neq \mathcal{U}(\omega_3)$, this can be rearranged to give

$$\mathcal{P}r(p) = \frac{\mathcal{U}(\omega_2) - \mathcal{U}(\omega_3)}{\mathcal{U}(\omega_1) - \mathcal{U}(\omega_3)}$$

As Ramsey notes, 'This amounts roughly to defining the degree of belief in p by the odds at which the subject would bet on p, the bet being conducted in terms of differences of value as defined' (pp. 179-80). This kind of *betting interpretation* of partial belief has been enormously influential within subjectivist approaches to probability, and is today often treated as the default or orthodox way to operationalise degrees of belief in many contemporary applications.

Ramsey's definition of conditional probability follows a similar tact:

> Suppose the subject indifferent between the options (1) ['ω_1 if q, ω_2 if $\neg q$'], (2) ['ω_3 if $p \,\&\, q$, ω_4 if $\neg p \,\&\, q$, and ω_2 if $\neg q$']. Then the degree of his belief in p given q is the ratio of the difference [in utility] between ω_1 and ω_4 to that between ω_3 and ω_4... (p. 180)

Given the assumed properties of $\mathcal{P}r$, $\mathcal{U}$, and how they combine to generate utilities for gambles, the stated indifferences can be shown to just in case

$$\mathcal{P}r(p \text{ given } q) \;=\; \frac{\mathcal{U}(\omega_1) - \mathcal{U}(\omega_4)}{\mathcal{U}(\omega_3) - \mathcal{U}(\omega_4)},$$

assuming $\mathcal{P}r(q) > 0$ and $\mathcal{U}(\omega_3) \neq \mathcal{U}(\omega_4)$.

Having outlined his measurement procedure, Ramsey notes that from his axioms and definitions, the following 'laws of probability' can be proven:

1. $\mathcal{P}r(p) + \mathcal{P}r(\neg p) \;=\; 1$
2. $\mathcal{P}r(p \text{ given } q) + \mathcal{P}r(\neg p \text{ given } q) \;=\; 1$
3. $\mathcal{P}r(p \,\&\, q) \;=\; \mathcal{P}r(p)\mathcal{P}r(q \text{ given } p)$
4. $\mathcal{P}r(p \,\&\, q) + \mathcal{P}r(p \,\&\, \neg q) \;=\; \mathcal{P}r(p)$

Finite additivity follows from the fourth of these. He goes on to state that:

> These are the laws of probability, which we have proved to be necessarily true of any consistent set of degrees of belief. Any definite set of degrees of belief which broke them would be inconsistent in the sense that it violated the laws of preference between options... If anyone's mental condition violated these laws, his choice would depend on the precise form in which the options were offered him, which would be absurd. He could have a book made against him by a cunning better and would then stand to lose in any event. (p. 182)

The reasoning behind the last claim is never made explicit.

It is dubious that Ramsey really proved that the mentioned laws must necessarily be true of any consistent set of partial beliefs—amongst the axioms of his representation theorem for utilities are several non-necessary conditions, and (moreover) his definitions are grounded in strong theoretical assumptions about the role partial beliefs play in generating preferences. Nevertheless, it is easy to see in Ramsey's argument an early version of what's come to be known as a Dutch Book Argument, later made more precise by de Finetti (1937; see entry on Dutch Book Arguments); and the foreshadowing of later representation theorem-based arguments for expected utility theory and probabilistic norms on belief (Skyrms 1987).

Bibliography

- Bradley, R., 2001, "Ramsey and the Measurement of Belief," in D. Corfield & J. Williamson (eds.), *Foundations of Bayesianism*, Dordrecht: Kluwer Academic Publishers, pp. 263–290.
- Davidson, D., Suppes, P., & Siegel, S., 1957, *Decision making; an experimental approach*, Stanford: Stanford University Press.
- Debreu, G., 1959, "Cardinal utility for even-chance mixtures of pairs of sure prospects," *The Review of Economic Studies*, 28 (3): 174–177.
- de Finetti, B., 1937, "La Prévision: Ses Lois Logiques, Ses Sources Subjectives," *Annales de l'Institut Henri Poincaré*, 7: 1-68; translated as "Foresight: Its Logical Laws, Its Subjective Sources," in H.E. Kyburg Jr. and H.E. Smokler (eds.), *Studies in Subjective Probability*, Huntington, NY: Robert E. Kreiger Publishing Co, 1980.
- Elliott, E., 2017, "Ramsey Without Ethical Neutrality," *Mind*, 126 (501): 1–51.
- Eriksson, L., & Hájek, A., 2007, "What Are Degrees of Belief?" *Studia Logica* (Special Issue, Formal Epistemology, Branden Fitelson (ed.)), 86 (2): 185–215.
- Eriksson, L., & Rabinowicz R., 2013, "The interference problem for the betting interpretation of degrees of belief," *Synthese*, 190: 809–830.
- Fishburn, P., 1967, "Preference-based definitions of subjective probability," *The Annals of Mathematical Statistics*, 38 (6): 1605–1617.
- —— 1981, "Subjective expected utility: A review of normative theories," *Theory and Decision*, 13 (2): 139–199.
- Jeffrey, R., 1965, *The Logic of Decision*, Chicago: University of Chicago Press.
- Ramsey, P.F., 1926, "Truth and Probability," in Ramsey & R.B. Braithwaite (ed.), *The Foundations of Mathematics and other Logical Essays*, London: Kegan, Paul, Trench, Trubner & Co., 1931, pp. 156–198.
- Sahlin, N., 1990, *The Philosophy of F.P. Ramsey*, Cambridge: Cambridge University Press.
- Savage, L.J., 1954, *The Foundations of Statistics*, New York: Wiley.
- Skyrms, B., 1987, "Coherence, in N. Rescher (ed.), *Scientific Inquiry in Philosophical Perspective*, Pittsburgh: University of Pittsburgh Press, pp. 225–242.
- Sobel, J.H., 1998, "Ramsey's Foundations Extended to Desirabilities," *Theory and Decision*, 44 (3): 231–278.