‘Ramseyfying’ Probabilistic Comparativism

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Abstract

Comparativism is the view that the fundamental doxastic state consists in comparative probability judgments (e.g., judging p to be more likely than q), with numerical degrees of belief (e.g., believing p to some numerical degree x) seen as theoretical tools used to represent and reason about systems of comparative judgements that satisfy a minimum standard of rationality. In this paper, I develop a version of comparativism inspired by a suggestion made by Frank Ramsey in his ‘Probability and Partial Belief’ (1929). I show how this ‘Ramseyan comparativism’ can be used to weaken the (unrealistically strong) conditions of probabilistic coherence that comparativism traditionally relies on.

1. Introduction

For those who deal in degrees of belief, a pressing issue concerns the theoretical basis of their numerical representation. It is typical to represent the strengths with which things can be believed with real numbers between 0 and 1, or (especially in recent years) with intervals of the reals. Moreover, it's typical to assume that these numbers encode more than merely *ordinal* information. For example, most would be perfectly happy to accept instances of the following as valid:

1. α believes p to degree x, and q to degree y.
2. $x = n \times y$.

$\therefore$ α believes p n times as much as she believes q.

This indicates a widespread commitment to the idea that strength of belief can be represented on a ratio scale, or at least something much like it. More generally, it marks a commitment to the idea that the numbers encode what we will call *cardinal* information.

And that's just the sort of commitment that ought to be explained (or explained away) by any minimally adequate account of what degrees of belief *are*. We don't get to posit cardinality for free; our beliefs don't come with little numbers literally attached to them. Rather, they must have some qualitative structure that is in some way or another similar to, and hence representable by,

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the real numbers in the unit interval—and in particular, such that the ordinal and at least some cardinal properties of and relations between those numbers (e.g., intervals, ratios, etc.) represent something doxastically meaningful. That's a strong claim, and it's not at all obvious what that structure *is*. So how is it that we manage to get from the purely qualitative stuff in the head through to numerically represented degrees of belief (henceforth: numerical degrees of belief) that encode interesting cardinal information?

There have been a few answers put forward to this question. One long-standing tradition seeks to explain "where the numbers come from" by considering how beliefs interact causally with *preferences* (e.g., Ramsey 1931). Others have tried to extract numerical degrees of belief out of *qualitative expectations*, a special kind of non-propositional attitude supposedly more fundamental than degrees of belief themselves. (See esp. Suppes and Zanotti 1976; Clark 2000.) Still other potential answers have yet to be fully explored. For instance, if you like the idea that degrees of belief are really just outright beliefs about objective probabilities, then you might think that whatever cardinality they possess is derivative upon the cardinal information possessed by those probabilities—wherever *that* comes from.

I'm inclined to think that all of the above are worth considering seriously, but in this paper I want to focus on one view in particular: *comparativism*. According to comparativists, the fundamental doxastic state consists in what we'll call *probability rankings*—that is, purely ordinal judgements concerning relative probabilities, such as the judgement that p is more probable than q, that r and s are equiprobable, or that t is at least as probable as u. With that as their starting point, the comparativist sees numerical degrees of belief as a kind of theoretical tool, a way to represent and reason about sufficiently rational systems of probability rankings. Or to put that another way: the numbers we use to represent our degrees of belief ultimately describe a purely ordinal structure imposed over a set of propositions by our probability rankings, when those rankings satisfy some minimum threshold of rationality.[1]

Prima facie, comparativism seems to struggle with providing any plausible explanation for cardinality. After all, individual probability rankings contain nothing more than ordinal information, so how could a system composed of nothing more than such rankings possess anything more than that?[2] Nevertheless, comparativists have what is by now a standard explanation of how numerical degrees of belief carry cardinal information. By drawing on a well-worn analogy with the measurement of mass, length, and other extensive quantities, comparativists have posited rationality conditions under which a kind of cardinal information might be extracted from a system of probability rankings.

As things currently stand, though, the rationality conditions to which comparativists typically appeal are quite strong indeed. Essentially, they impose a qualitative form of probabilistic coherence on any agent who can be said to have numerical degrees of belief at all. And this is a key limitation with the view in its typical form: it lacks an adequate account of numerical degrees of belief for agents more realistically construed—that is, agents who might fail to live up to

[1] Comparativism has been around for a long time, and is still influential today. We find it in one form or another in, *inter alia*, (deFinetti 1931), (Koopman 1940a), (Kraft et al. 1959), (Fine 1973, pp. 68ff), (Krantz et al. 1971, p. 200), (Fishburn 1986), (Hawthorne 2016) and (Stefánsson 2017).

[2] For a recent complaint along just these lines, see (Meacham and Weisberg 2011, p. 659).

the very strict standards of probabilistic coherence. Consequently, in this paper I want explore whether, and how, the standard rationality conditions can be weakened, so as to widen the scope of the comparativist's theory.

I will begin my discussion by reviewing the standard account of how mass is measured on a ratio scale (§2), and then I'll look at how probabilistic comparativism posits an essentially similar process for the measurement of degrees of belief (§3). Following that, I'll discuss in more detail the motivations for seeking a more general basis for comparativism (§4). Finally, I will show that the strong rationality conditions of traditional *probabilistic comparativism* can indeed be weakened to a significant extent—though not without limits. I will do this by developing what I call *Ramseyan comparativism* (§5). Moreover, I will show that the Ramseyan conditions on probability rankings are in one important respect *maximally* weak: inasmuch as comparativists want to retain the analogy with the measurement of mass as that analogy is usually understood, Ramseyan comparativism is as general as it gets.

2. The Measurement of Mass

Let a and b be any two concrete objects you like, and compare:

> Ordinal. a is more massive than b.
> Cardinal. a is twice as massive as b.

Cardinal obviously contains strictly more information than Ordinal, and that information has to come from somewhere. Yet masses don't come with little numbers literally attached to them; no physical quantity does. Whatever it is that explains the extra information in Cardinal must ultimately be purely qualitative in nature. So how might we get from the purely qualitative facts on the ground through to numerical masses that encode interesting cardinal information?

The representational theory of measurement gives us a plausible answer.[3] First, note that Cardinal is true (roughly) if and only if, if you were to take two disjoint objects each as massive as b (call them b_1 and b_2, b's *duplicates*) and join them together, then the resulting object would be just as massive as a. Call the operation of joining objects together *concatenation*; we assume that no mass is gained or lost in the act of concatenating. Given this, it's plausible that there's nothing more to the truth of a claim like Cardinal than what we've just said—that is, 'a is twice as massive as b' just means something roughly to the effect of 'a is as massive as the concatenation of two duplicates of b.' By reference, then, to purely ordinal comparisons between duplicates and the concatenations thereof, we've been able to give straightforward *qualitative* meaning to Cardinal.

And we can easily generalise this idea to explain rational ratio comparisons more generally. For positive integers n, m, say that a is ${}^{n}/_{m}$ times as massive as b whenever there's some object c such that

(i) a is as massive as the concatenation of n duplicates of c, and
(ii) b is as massive as the concatenation of m duplicates of c.

[3] The *locus classicus* for this theory is (Krantz et al. 1971). For an early and very readable introduction, see (Suppes and Zinnes 1963); and for detailed treatments aimed at a philosophical audience, see (Swoyer 1991) and (Matthews 2007, ch. 5).

Now let x designate c's mass in whatever units you like—let's go British imperialist and say *slugs* (about 14.6 kg). Intuitively, a must then have a mass of $n \times x$ slugs, and b must have a mass of $m \times x$ slugs. Hence, a is $^{n}/_{m}$ times as massive as b. Indeed, with a bit more work, and a qualitative version of the method of Dedekind cuts, we can generalise the idea even further to explain arbitrary *real* ratio comparisons; however, for the sake of simplicity we'll stick with rational ratios for the rest of this discussion.

Hiding in the background here is a crucial empirical assumption: that the operation of concatenation behaves as a kind of qualitative analogue of *addition*. We rely on exactly this assumption to move from, e.g., 'a is as massive as the concatenation of n duplicates of an object with a mass of x slugs' to 'a has a mass of $n \times x$ slugs'—i.e., we assume that the mass of a concatenation is just the sum of the masses of the concatenands. (Imagine if, instead, concatenation behaved like quaddition: whenever you concatenate up to 57 duplicates together, things are as usual; but concatenate more and the result is always as massive as 5 duplicates. We could have then used concatenations to define our way up to one object's being 57 times as massive than another, but no further.)

Fortunately for us, the analogy between concatenation and addition is quite close.[4] Where

$$\begin{aligned} a \succsim^{m} b \ &\text{ iff }\ a \text{ is at least as massive as } b, \\ a \sim^{m} b \ &\text{ iff }\ a \text{ is exactly as massive as } b, \\ a \oplus b \ &= \text{ the concatenation of } a \text{ and } b, \end{aligned}$$

then it's plausible that $\succsim^{m}$ is transitive and complete, and $\sim^{m}$ is just the symmetric part of $\succsim^{m}$. Furthermore, $\oplus$ behaves with respect to $\succsim^{m}$ a lot like $+$ behaves with respect to $\geq$. In particular, for all disjoint objects a, b, c,

1. $a \oplus b \succsim^{m} b$. (*positivity*)
2. $a \oplus b \sim^{m} b \oplus a$. (*commutativity*)
3. $a \oplus (b \oplus c) \sim^{m} (a \oplus b) \oplus c$. (*associativity*)
4. $a \succsim^{m} b$ iff $a \oplus c \succsim^{m} b \oplus c$. (*monoticity*)

Now compare these with the following essential properties of $+$ with respect to $\geq$, where n and m are non-negative real numbers:

1. $n + m \geq m$. (*positivity*)
2. $n + m = m + n$. (*commutativity*)
3. $n + (m + k) = (n + m) + k$. (*associativity*)
4. $n \geq m$ iff, for any k, $n + k \geq m + k$. (*monotonicity*)

Indeed, if we posit a rich enough space of concrete objects and make one further 'Archimedean' assumption—roughly: that no object is infinitely more massive than any other—then we can say something stronger still: where $\boldsymbol{\mathcal{O}}$ is the set of concrete objects and $\mathbb{R}^{+}$ the positive reals, the relational system $< \boldsymbol{\mathcal{O}}, \succsim^{m}, \oplus >$ has (essentially) the same formal structure as $< \mathbb{R}^{+}, \geq, + >$. Thus, we can assign numbers to objects in such a way that $\succsim^{m}$ is precisely modelled by $\geq$, and $\oplus$ is precisely modelled by $+$. And with that in hand, we

[4] At least, the analogy is very close in Newtonian physics. Things are different at very small scales in special and general relativity theory, where there can be a difference (the mass defect) between the mass of a composite and the sum of the masses of its parts. See (McQueen 2015) for discussion.

can start to define up ratios of masses, numerical differences in mass, ratios of differences in mass, and so on—i.e., we have the basic resources needed to explain how our assignments of numerical masses manage to carry all sorts of interesting cardinal information.

The upshot: numerical masses represent a fully qualitative system of ordinal mass comparisons which have an 'additive' structure over concatenations. We're justified in treating ratios of masses as meaningful because there exists an operation on objects that is intuitively and formally like 'adding' masses together. And we can apply the same basic idea outlined here to account for the measurement of other (extensive) quantities. For instance, a is *twice as long as* b iff a is as long as two length-duplicates of b laid end-to-end; a has *twice the volume of* b iff a has the same volume as two volume-duplicates of b joined together; and an event E_1 has *twice the duration* of E_2 iff E_1 can be split into two disjoint events with the same duration as E_2.

To apply the same idea to account for numerical degrees of belief, comparativists have historically sought to identify an operation on the relata of a subject's probability rankings (i.e., propositions) that behaves, with respect to those judgements, similarly enough to addition to justify treating it as a qualitative analogue thereof. As Krantz et al. put it, the strategy is 'to treat the assignment of [subjective] probabilities as a measurement problem of the same fundamental character as the measurement of, e.g., mass or duration' (1971, p. 200). So let's see how that plays out in practice.

3. Probabilistic Comparativism

In this section I'll provide a detailed review of probabilistic comparativism. I'll begin by laying out some basic notation and assumptions (§3.1), followed by the mathematical underpinnings of the view (§3.2). Finally, I'll define two specific varieties of probabilistic comparativism—one 'precise' (§3.3), and the other 'imprecise' (§3.4).

3.1 Notation and assumptions

For the remainder of this paper, let α be an arbitrary thinking subject, whose numerical degrees of belief we are trying to explain. I assume that the propositions regarding which α has beliefs can be modelled as subsets of some space of logically possible worlds, Ω. By 'logically possible', I mean no more than that the worlds are closed under a consequence relation at least as strong as that of classical propositional logic. So, you can assume that Ω includes metaphysically or even epistemically impossible worlds, if that's what floats your boat—as long as the worlds are classically logically consistent. I'll talk more about this assumption in §4.

Next, let $\mathcal{B} \subseteq \wp(\Omega)$ denote that set of propositions regarding which α has probability rankings and/or numerical degrees of belief. For simplicity, I'll assume throughout that $\mathcal{B}$ is an algebra of sets on Ω; in other words, $\mathcal{B}$ contains Ω, and it's closed under relative complements and binary intersections/unions.

Given that, I'll assume that α's full system of probability rankings can be modelled with a single binary relation $\succsim$ defined over $\mathcal{B}$, where

$$p \succsim q \text{ iff } \alpha \text{ believes } p \text{ at least as much as she believes } q.$$

Consequently, where $\succ$ and $\sim$ stand for the comparatives *more probable* and *equally probable* respectively, I am in effect assuming that

$$p \sim q \text{ iff } (p \succsim q) \,\&\, (q \succsim p), \qquad p \succ q \text{ iff } (p \succsim q) \,\&\, \neg(q \succsim p).$$

Nothing about this last assumption should be treated as obvious or trivial. For example, α might think that p is at least as likely as q, without thinking that p is more likely than q, or that p is just as likely as q. Nevertheless, the assumption will simplify the discussion considerably, and nothing of great importance will hang on it.

Finally, where a *credence function* $\mathcal{C}r$ assigns real numbers to the propositions in $\mathfrak{B}$, I'll say that $\mathcal{C}r$ *almost agrees with* $\succsim$ iff, for all relevant propositions p, q,

$$p \succsim q \text{ only if } \mathcal{C}r(p) \geq \mathcal{C}r(q);$$

and we'll say that $\mathcal{C}r$ *agrees with* $\succsim$ just in case

$$p \succsim q \ \text{ iff } \ \mathcal{C}r(p) \geq \mathcal{C}r(q).$$

For ease of expression, I'll treat *agreement* (but not *almost agreement*) as symmetric: $\succsim$ agrees with $\mathcal{C}r$ just in case $\mathcal{C}r$ agrees with $\succsim$.

3.2 Agreeing with probabilities

Any credence function that agrees with a probability ranking $\succsim$ is *ipso facto* at least an ordinal-scale representation of $\succsim$. Our task now is to lay out conditions under which a credence function can be said to also carry cardinal information. This is where probability functions can come in handy:

Definition 1. $\mathcal{C}r : \mathfrak{B} \mapsto \mathbb{R}$ is a *probability function* iff, $\forall p, q \in \mathfrak{B}$,
(i) $\mathcal{C}r(\Omega) = 1$,
(ii) $\mathcal{C}r(p) \geq 0$, and
(iii) If $p \cap q = \varnothing$, then $\mathcal{C}r(p \cup q) = \mathcal{C}r(p) + \mathcal{C}r(q)$.

It follows immediately from criterion (iii) that *if* some probability function—any probability function—agrees with $\succsim$, then the union of disjoint sets is to $\succsim$ just as $\oplus$ is to $\succsim^m$, or as $+$ is to $\geq$. Great! That's exactly the kind of thing that's needed to found the analogy with mass.

Moreover, we have known for a long time the exact conditions under which a probability ranking will agree with some probability function. Where $\boldsymbol{\mathcal{B}}$ is finite, the following five conditions are individually necessary and jointly sufficient (see Scott 1964). For all $p, q, r \in \boldsymbol{\mathcal{B}}$,[5]

C1. $p \succsim q$ or $q \succsim p$. (*completeness*)
C2. $p \succsim p$. (*reflexivity*)
C3. If $p \succsim q$ and $q \succsim r$, then $p \succsim r$. (*transitivity*
C4. $\Omega \succ \varnothing$. (*non-triviality*)
C5. $p \succsim \varnothing$. (*non-negativity*)

[5] In the context of the other conditions, C2 and C3 are redundant, and C6 is equivalent to the slightly weaker formulation of Scott's axiom found in (Scott 1964). I've stated the conditions in this way because it will make the discussion easier later on.

C6. Where $\mathbf{1}_p$ denotes the indicator function of p, $(p_i)_{i=1}^n$ and $(q_i)_{i=1}^n$ are finite sequences of propositions, and $(k_i)_{i=1}^n$ is a finite sequence of natural numbers, then if

(i) $\sum_{i=1}^n k_i \cdot \mathbf{1}_{p_i}(\omega) = \sum_{i=1}^n k_i \cdot \mathbf{1}_{q_i}(\omega)$ for all $\omega \in \Omega$, and
(ii) $p_i \succsim q_i$, for $i = 1, \ldots, n-1$,

then $q_n \succsim p_n$. (*Scott's axiom*)

For this reason, comparativists have frequently suggested that, *at least when* $\succsim$ *satisfies the conditions* C1–C6, partial beliefs can be represented on a ratio scale, with the union of disjoint sets playing the role of concatenation.

However, it is possible to say something more general than this, and doing so will be useful in demonstrating continuity between probabilistic comparativism and the Ramseyan comparativisms discussed below. First, note that criterion (iii) of Definition 1 also implies:

(iv) If $\mathcal{C}r(p \cap q) = 0$, then $\mathcal{C}r(p \cup q) = \mathcal{C}r(p) + \mathcal{C}r(q)$.

That is, probability functions are also additive with respect to the union of what we might call *pseudodisjoint* propositions, where p and q are pseudodisjoint for α just in case she has zero confidence in their intersection, $p \cap q$. Or, more formally,

Definition 2. p is *minimal* iff $q \succsim p$ for all $q \in \boldsymbol{\mathcal{B}}$, and p is *maximal* iff $p \succsim q$ for all $q \in \boldsymbol{\mathcal{B}}$.

Definition 3. $\boldsymbol{\mathcal{P}} \subseteq \boldsymbol{\mathcal{B}}$ is a *set of pseudodisjoint propositions* iff, for any $\boldsymbol{\mathcal{P}}^\star \subseteq \boldsymbol{\mathcal{P}}$ where $|\boldsymbol{\mathcal{P}}^\star| \geq 2$ and any minimal q, $\bigcap \boldsymbol{\mathcal{P}}^\star \sim q$; furthermore, propositions $p_1, \ldots, p_n$ are *pairwise pseudodisjoint* iff there's a set of pseudodisjoint propositions $\boldsymbol{\mathcal{P}}$ such that $p_1, \ldots, p_n \in \boldsymbol{\mathcal{P}}$.

That is, assuming that α has exactly zero confidence in p whenever p is minimal, Definition 3 plausibly characterises in comparative terms what it is for α to think that at most one proposition from $p_1, \ldots, p_n$ is true.[6]

With all that in hand, we can note that C1–C6 together imply that $\succsim$ is Archimedean and, where p, q, r are pairwise pseudodisjoint,

1. $(p \cup q) \succsim q$ (*positivity*)
2. $(p \cup q) \sim (q \cup p)$ (*commutativity*)
3. $(p \cup (q \cup r)) \sim ((p \cup q) \cup r)$ (*associativity*)
4. $p \succsim q$ iff $(p \cup r) \succsim (q \cup r)$ (*monotonicity*)

Again, this is exactly what comparativists need in order to draw the analogy with the measurement of mass.

3.3 Precise probabilistic comparativism

So let's turn the foregoing mathematical points into some philosophical hypotheses. Assuming that $\mathcal{C}r$ agrees with α's probability ranking, say that $\mathcal{C}r$ constitutes a *fully adequate model* of α's degrees of belief whenever

$$\alpha \text{ believes } p \; {}^{n}\!/_{m} \text{ times as much as she believes } q \text{ iff } \mathcal{C}r(p) = \frac{n}{m} \times \mathcal{C}r(q).$$

[6] Definition 3 implies that every singleton set $\{p\} \in \boldsymbol{\mathcal{B}}$ is trivially a 'set of pseudodisjoint propositions.' This is a feature, not a bug. The rather tortured definition will be useful later on, when we move away from probability functions.

I assume that full adequacy is what comparativists ought to strive for—after all, most theorists are willing to accept the validity of *both* of the following kinds of inferences:

1. α believes p to degree x, and q to degree y.
2. $x = n \times y$.

$\therefore$ α believes p n times as much as she believes q.

and in the other direction,

1. α believes p n times as much as q.
2. α believes p to degree y.

$\therefore$ α believes q to degree $x = n \times y$.

Only full adequacy licenses both of these directions, and so I take it that full adequacy stands as a basic desideratum for any comparativist theory. Nevertheless, we can also say that $\mathcal{C}r$ is *L-to-R adequate* just in case the left-to-right direction of the above biconditional holds, and *R-to-L adequate* just in case the right-to-left direction holds. Individual comparativists may well want to reject full adequacy in favour of mere L-to-R or R-to-L adequacy, provided that the rejection is well-motivated and they are able to explain away the intuitions in favour of full adequacy. (I'll say a little more about this in §5.2.)

So given that, let *precise probabilistic comparativism* denote any comparativist theory that's committed to the following conditional:

> Precise Probabilistic Comparativism. If $\mathcal{C}r$ is the unique probability function that agrees with α's probability ranking, then $\mathcal{C}r$ is a fully adequate model of α's degrees of belief.

Note the requirement that the probability function be *unique*. This is necessary to avoid contradiction. For any non-trivial algebra $\boldsymbol{\mathcal{B}}$, there will always be some probability functions on $\boldsymbol{\mathcal{B}}$ that agree with the very same probability ranking. And since any two probability functions on the same domain will disagree on at least some ratios, a general pattern of inference from '$\mathcal{C}r(q) = {}^{n}\!/_{m} \times \mathcal{C}r(q)$' to '$\alpha$ believes p ${}^{n}\!/_{m}$ times as much as q' will be valid *only* when the $\mathcal{C}r$ is unique in the relevant sense. In other words, R-to-L adequacy presupposes uniqueness, which in turn requires further conditions on $\succsim$.

There are several conditions that suffice to establish uniqueness. Of particular note is the following, which Stefánsson (2017, 2018) uses to ensure uniqueness in his recent defences of probabilistic comparativism:

> Continuity. For all non-minimal p, q, there are p', q' such that $p \sim p'$, $q \sim q'$, and p' and q' are each the union of some subset of a finite set of disjoint propositions $\{r_1, \ldots, r_n\}$ such that $r_i \sim r_j$ for $i, j = 1, \ldots, n$.

The interested reader can see (Krantz et al. 1971, §5.2) and (Fishburn 1986) for a range of other conditions sufficient to ensure uniqueness.

Now, probabilistic comparativism clearly has some resources to put forward an account of how our numerical degrees of belief carry cardinal information, *in the event that* $\succsim$ *satisfies the requisite conditions.* In particular, consider the following principle, which is the comparative probability judgement version of how we defined rational ratio comparisons for mass in §2:[7]

[7] The first clause of the General Ratio Principle is a close relative of Stefánsson's (2018) 'Ratio Principle.' The second (inductive) clause is new—in the context of a condition like Continuity it's redundant, but see §4 for it put to work.

General Ratio Principle. α believes p $^{n}/_{m}$ times as much as q if

(i) For $0 < n \leq m$, there are m non-minimal, equiprobable pairwise pseudodisjoint propositions $r_1, \ldots, r_m$ such that $q \sim (r_1 \cup \cdots \cup r_m)$ and $p \sim (r_1 \cup \cdots \cup r_n)$; or

(ii) α believes p $^{n'}/_{m'}$ times as much as r, and believes r $^{n''}/_{m''}$ times as much as q, where $^{n}/_{m} = {}^{n'\cdot n''}/_{m'\cdot m''}$.

So, for instance, α will take p to be twice as probable as q if there is some proposition q' that's obviously inconsistent with q such that $q \sim q'$ and $(q \cup q') \sim p$. In this case, q and q' are acting as 'duplicates' of one another, and $q \cup q'$ is their 'concatenation'.

3.4 Imprecise probabilistic comparativism

Say that $\mathcal{C}r$ *coheres* with the GRP just in case, whenever that principle implies that p is believed $^{n}/_{m}$ times as much as q, then $\mathcal{C}r(p) = {}^{n}/_{m} \times \mathcal{C}r(q)$; otherwise, it *conflicts* with the GRP. Interestingly, if any probability function *almost* agrees with $\succsim$ and $\varnothing$ is minimal, then that function coheres with the GRP. This means that it's possible to extend the account of ratio comparisons just given to imprecise probabilities and incomplete probability rankings.

For non-ideal agents, the completeness condition (C1) is widely considered highly implausible. We should expect plenty of gaps in $\succsim$. Consider the following example, adapted from (Fishburn 1986):

p = The global population in 2100 will be greater than 13 billion.
q = The next card drawn from this old and incomplete deck will be a heart.

p and q are sufficiently far removed from one another that it's hard to make a judgement as to which is more likely than the other. Similar examples abound.

There's a natural way of dealing with incompleteness to which comparativists can (and do) appeal. Where $\mathfrak{F}$ is any set of real-valued functions on $\wp(\Omega)$, say this time that the *set* $\mathfrak{F}$ agrees with $\succsim$ just in case for all relevant p, q,

$$p \succsim q \text{ iff } \mathcal{C}r(p) \geq \mathcal{C}r(q) \text{ for all } \mathcal{C}r \in \mathfrak{F}.$$

The idea behind a set-of-functions model is to recapture the probability ranking by supervaluating over the functions in $\mathfrak{F}$—only what's common to every such is treated as having representative import. Whenever $\succsim$ fails to hold between p and q, $\mathfrak{F}$ will contain at least one pair of probability functions that disagree on the relative ordering of p and q; hence, we manage to 'numerically' represent $\succsim$.

Where $\mathcal{B}$ is finite, a set of probability functions agrees with $\succsim$ just in case the latter satisfies C2–C6 (see Alon and Lehrer 2014). Furthermore, while there will often be more than one set of probability functions $\mathfrak{F}$ that agrees with $\succsim$, the union of all such sets will always agree with $\succsim$. So, whenever $\succsim$ satisfies C2–C6 there's always a unique set of probability functions that agrees with $\succsim$ that's maximal with respect to inclusion.

Hence, if we extend the definitions of full / L-to-R / R-to-L adequacy in the natural way (i.e., by inserting '$\forall \mathcal{C}r \in \mathfrak{F}$' in the appropriate locations), we can characterise *imprecise probabilistic comparativism* by its commitment to:

Imprecise Probabilistic Comparativism. If a non-empty set of probability functions $\mathfrak{F}$ agrees with α's probability ranking and $\mathfrak{F}$ is maximal with respect to inclusion, then $\mathfrak{F}$ is a fully adequate model of α's degrees of belief.

Imprecise probabilistic comparativism implies the precise version. That is, if we assume that $\mathcal{F}$ and $\mathcal{C}r$ are essentially the same model whenever $\mathcal{F} = \{\mathcal{C}r\}$, then the two varieties of comparativism say exactly the same thing whenever exactly one probability function agrees with $\succsim$.

Furthermore, every $\mathcal{C}r$ in a set $\mathfrak{F}$ that agrees with $\succsim$ will itself *almost* agree with $\succsim$. So, if we also extend the definition of *coherence* in the natural way to sets of functions, it follows that if a set of probability functions $\mathfrak{F}$ agrees with $\succsim$, then $\mathfrak{F}$ coheres with the GRP. In short, both the precise and imprecise versions of probabilistic comparativism explain cardinal comparisons using *basically* the same principles.

4. Why Generalise?

We've seen now that C2–C6 are sufficient for the union of pseudodisjoint sets to behave like addition, but they are by no means necessary. It is possible to weaken those conditions while maintaining the analogy, and it is of significant importance for comparativism that this can be done. C6, in particular, is a very strong condition indeed, especially in the context of the (individually weak) conditions C4 and C5. It's the kind of condition that we could only reasonably expect to be satisfied by a highly idealised agent, and that severely limits the applicability of probabilistic comparativism as an account of how *we* mere ordinary agents might come to have numerical degrees of belief that carry rich cardinal information.

For example, it's a consequence of C4–C6 that if $p \subseteq q$ and $p, q \in \mathfrak{B}$, then $p \precsim q$. This immediately gives rise to a probabilistic version of the problem of logical omniscience. If the worlds in Ω are closed under any consequence relation $\Rightarrow$ whatsoever, then for all $p, q \in \mathfrak{B}$, it follows that if $p \Rightarrow q$, then $p \precsim q$. That is, any probability ranking that is (i) defined over propositions taken from a space of worlds that's closed under $\Rightarrow$, and (ii) agrees with a (set of) probability function(s), is *ipso facto* necessarily coherent with respect to $\Rightarrow$. In §3.1 it was assumed that $\Rightarrow$ is at least as strong as the relation we find in classical propositional logic, and it's deeply implausible that ordinary agents' probability rankings are everywhere and always coherent with respect to *that* logic. But the point can be put more generally: unless $\Rightarrow$ is *extremely* weak, it's certainly plausible that the probability rankings of any ordinary agents might falsify at least one of those conditions C4 through C6.

And it would be unreasonable to say that an agent α doesn't have degrees of belief just because she's not ideally rational, or that the satisfaction of a very strong rationality condition like C6 is a necessary precondition for having numerical degrees of belief that encode interesting cardinal information. That would be manifestly implausible: even if she were *highly irrational*, α could still believe one proposition *much more* than she believes another, or *at least twice as much* as she believes another.[8] This should be uncontroversial—only someone caught firmly in the grips of an unrealistic picture of belief would think to deny it. Our capacity to believe something much more than another thing, or (at least) n times as much as another thing, is not hostage to any presupposition of idealised rationality. And an explanation of cardinality that works *only* in the

[8] The point here is independent of the matter of how *precise* the partial beliefs of ordinary agents are. Even if α's degrees of belief were everywhere imprecise, she could still believe p at least twice as much as q.

ideal case is, at best, incomplete—and at worst, no explanation at all. All else being equal, it would be better to have an account of cardinality that applies equally well to the lowest common denominator.

This is *not* to deny the obvious point that it's often useful to get an explanation of some phenomenon working for an idealised model before moving on to less ideal cases. That is how science works in general, and it's exactly how we should expect things to work here. But an idealised model does real-world explanatory work only to the extent that it does not depend critically on the idealisations in question. Models have explanatory value when the conclusions we can draw from them are robust under variations to their idealising conditions; they should not break down when realism is added back in. In the present case, then, it would be useful to have some assurance that the usual comparativist account of cardinality does not depend critically on unrealistic assumptions.

But before I discuss how the comparativist might generalise away from C4–C6, in the remainder of this section let me briefly deal with one important response to the foregoing points. In particular, you might think that *impossible worlds* could help to avoid the above worries about logical omniscience.

In a little more detail, the objection is this. My argument that C4–C6 are too strong assumes that Ω is closed under a relatively strong consequence relation $\Rightarrow$. But, where Ω is a rich enough space of *and* impossible worlds, it's well-known that we can construct probability functions that 'mimic' the behaviour of *any* non-probabilistic function on $\boldsymbol{\mathcal{B}}$.[9] In other words, what looks like numerical degrees of belief that are inconsistent with C4–C6 when they're defined for propositions *qua* sets of merely possible worlds, can in fact be *re*-modelled using (sets of) probability functions, *if* we help ourselves to enough impossible worlds. For the very same reason, what seemed like very strong rationality conditions only satisfiable by idealised agents, C4–C6, could instead be seen as fully compatible with the belief rankings of even very non-ideal agents. Hence, to generalise the comparativist explanation of cardinality, we don't need to generalise the conditions imposed on $\succsim$—we just need to make sure that there are enough impossible worlds in Ω.

I do not think that this is a viable strategy for the comparativist to adopt. I'll set out the reasons for this very briefly, since the relevant issues are discussed at length in (Elliott forthcoming). The problem is that once Ω includes enough impossible worlds for the strategy to work—roughly, for any impossibility, there's an impossible world that verifies it—then most subsets of Ω will be *meaningless*, not representative of any proper contents of thought. Moreover, the set of *meaningful* subsets of Ω will have precisely zero interesting set-theoretic structure. For any meaningful subset p of Ω, none of p's subsets or supersets will be meaningful, and nor will any subset of $\Omega \setminus p$ be meaningful. In short, having too many impossible worlds in Ω renders useless any set-theoretic definition of 'concatenation'.[10] More directly, any algebra of propositions defined on a space of possible and impossible worlds that's minimally rich enough to

[9] More precisely: if $\boldsymbol{\mathcal{B}}$ is a countable subset of $\wp(\Omega)$, where Ω is a space of possible worlds, then if Ω^+ is a rich enough extension of Ω (i.e., has enough impossible worlds), and $\mathcal{C}r$ is any function from $\boldsymbol{\mathcal{B}}$ into $[0, 1]$, then there's a probability function $\mathcal{C}r^+$ on an appropriate algebra of sets $\boldsymbol{\mathcal{B}}^+$ over Ω^+ such that $\mathcal{C}r^+$ assigns x to the subset of Ω^+ that verifies φ just in case $\mathcal{C}r$ assigns x to the subset of Ω that verifies φ. See (Cozic 2006), (Halpern and Pucella 2011), and (Elliott forthcoming).

[10] Of course, we don't *have* to define 'concatenations' set-theoretically. But the only other place we'll plausibly find the requisite structure is in the *logical* relations amongst the contents

represent the contents of belief will contain only meaningful propositions just when the relevant space of worlds is closed under a consequence relation that is, for all intents and purposes, at least as strong as classical propositional logic.

So the probabilistic comparativist faces a dilemma: stick with possible worlds and face the problems of logical omniscience that come with it, or try to use impossible worlds but remove any means of defining 'concatenations' set-theoretically. Neither horn looks feasible. Impossible worlds aren't a magical solution to comparativism's generalisability worries—quite the opposite.

5. The Ramseyan Alternatives

If comparativism is to stand any chance of explaining what numerical degrees of belief are and how they carry cardinal information, it needs that the basic form of that explanation can be generalised away from the requirements of probabilistic coherence. That's where Ramseyan comparativism can help.

Ramseyan comparativism is inspired by a brief remark from Frank Ramsey's 'Probability and Partial Belief':

> [...] 'Well, I believe it to an extent $^2/_3$', i.e. (this at least is the most natural interpretation) 'I have the same degree of belief in it as in $p \vee q$ when I think p, q, r equally likely and know that exactly one of them is true'. (1929, p. 256)

In a recent paper, Weatherson (2016, pp. 223–4) has also suggested that Ramsey's remark points towards a version of comparativism that's weaker than probabilistic comparativism. However, neither Ramsey nor Weatherson take their discussion beyond this initial suggestion, and (as we'll see) there's a bit of work that needs to be done to flesh the idea out in full.

In the remainder of this paper, I will develop precise Ramseyan comparativism (§5.1, and then an imprecise version (§5.2). Following that, I will prove an important result about the conditions under which Ramseyan comparativism supports the analogy with the measurement of mass (§5.3). I also very briefly describe how Ramsey's idea might be developed along slightly different lines in an Appendix.

5.1 Precise Ramseyan comparativism

First, we will need another definition:

Definition 4. A set of n pseudodisjoint propositions $\mathcal{P}$ is an *n-scale of p* iff $\bigcup \mathcal{P} \sim p$ and for all q, q' in $\mathcal{P}$, $q \sim q'$.

We can take this as a comparativist characterisation of what it is for an agent to think that q's as likely as a disjunction of equiprobable propositions at most one of which is true. So, e.g., if α thinks q is as likely as $p_1 \cup p_2$, where p_1 and p_2 are equiprobable and pseudodisjoint, then $\{p_1, p_2\}$ is a 2-scale of q. We'll also need to assume that α is *certain* of p's truth just in case p is maximal. This is non-trivial, but it also appears to be unavoidable given the limited resources with which *certainty* might be defined within the present framework.

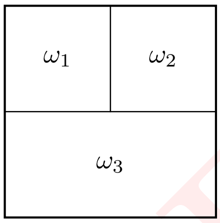
that the meaningful subsets of Ω^+ represent. That is, we could define 'concatenations' in terms of the *disjunctions of inconsistent contents*. But doing things this way brings us right back to where we started *vis-à-vis* the generalisability issues associated with C4–C6.

In the terms of Definition 4, Ramsey's idea becomes: α believes p to degree $^n/_m$ if $p \sim (q_1 \cup \cdots \cup q_n)$, where the $q_1, \ldots, q_n$ belong to an m-scale $\{q_1, \ldots, q_n, \ldots, q_m\}$ of some maximal proposition r. A good start—but there's a natural extension that will be helpful to incorporate into what follows.

Consider the following situation: $\boldsymbol{\mathcal{B}}$ is the powerset of $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\succsim$ is transitive, and

$$\Omega \succ \{\omega_1, \omega_3\} \sim \{\omega_2, \omega_3\} \succ \{\omega_1, \omega_2\} \sim \{\omega_3\} \succ \{\omega_1\} \sim \{\omega_2\} \succ \varnothing$$

We can represent $\succsim$ as follows, where the relative sizes of the boxes containing the ω_i correspond to the order of propositions in the probability ranking:

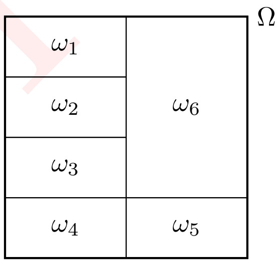


Now $\{\Omega\}$ is a 1-scale of Ω, and $\{\{\omega_3\}, \{\omega_1, \omega_2\}\}$ is a 2-scale of Ω, so Ramsey would say that

$$\mathcal{C}r(\Omega) = 1, \quad \mathcal{C}r(\{\omega_3\}) = \mathcal{C}r(\{\omega_1, \omega_2\}) = {}^1/_2.$$

However, the singletons $\{\omega_1\}$ and $\{\omega_2\}$ don't belong to any n-scale of Ω, so Ramsey's idea doesn't yet give us any strength with which they're believed. But since $\{\{\omega_1\}, \{\omega_2\}\}$ is a 2-scale of $\{\omega_1, \omega_2\}$, it's only reasonable to say that $\mathcal{C}r(\{\omega_1\}) = \mathcal{C}r(\{\omega_2\}) = {}^1/_4$.

Likewise, consider the following case, where $\Omega = \{\omega_1, \ldots, \omega_6\}$:



Here, assume that Ω is maximal and $\varnothing$ minimal, and $\succsim$ includes:

$$\{\omega_5, \omega_6\} \sim \{\omega_1, \omega_2, \omega_3, \omega_4\} \succ \{\omega_6\} \sim \{\omega_1, \omega_2, \omega_3\} \succ \{\omega_1\}$$

$$\{\omega_1\} \sim \{\omega_2\} \sim \{\omega_3\} \sim \{\omega_4\} \sim \{\omega_5\}$$

This time, $\{\{\omega_5, \omega_6\}, \{\omega_1, \omega_2, \omega_3, \omega_4\}\}$ is a 2-scale of Ω, and $\{\omega_1\}, \{\omega_2\}, \{\omega_3\}$ are three members of the 4-scale $\{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}$ of $\{\omega_1, \omega_2, \omega_3, \omega_4\}$; we'd therefore like to say that $\mathcal{C}r(\{\omega_1, \omega_2, \omega_3\}) = {}^3/_4 \times {}^1/_2 = {}^3/_8$. We note also that $\{\{\omega_6\}\}$ is a 1-scale of $\{\omega_1, \omega_2, \omega_3\}$; hence, $\mathcal{C}r(\{\omega_6\}) = {}^3/_8$.

We can capture the foregoing points by means of the following definition:

Definition 5. For integers n, m such that $m \geq n \geq 0$, $m > 0$, p is

(i) ${}^{0}/_{m}$*-valued* if p is minimal and ${}^{m}/_{m}$*-valued* if p is maximal, and

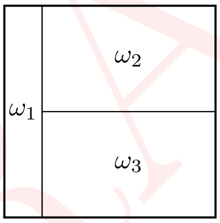
(ii) ${}^{n}/_{m}$*-valued* if $p \sim (q_1 \cup \cdots \cup q_{n'})$, where the $q_1, \ldots, q_{n'}$ belong to an m'-scale of an ${}^{n''}/_{m''}$-valued proposition, and ${}^{n' \cdot n''}/_{m' \cdot m''} = {}^{n}/_{m}$.

The generalised version of Ramsey's suggestion now amounts to the claim that α believes p to degree ${}^{n}/_{m}$ if p is ${}^{n}/_{m}$-valued. As such, define a *Ramsey function* as follows:

Definition 6. $\mathcal{C}r : \boldsymbol{\mathcal{B}} \mapsto [0,1]$ is a *Ramsey function* (relative to $\succsim$) iff, for all $p \in \boldsymbol{\mathcal{B}}$, if p is ${}^{n}/_{m}$-valued, then $\mathcal{C}r(p) = {}^{n}/_{m}$.

The connection between Ramsey functions and the GRP should be immediately apparent. In fact, in present terminology, the first (non-inductive) clause of the GRP states that for $m \geq n$, p is believed ${}^{n}/_{m}$ times as much as q whenever $\boldsymbol{\mathcal{P}}$ is an m-scale of q, and $\boldsymbol{\mathcal{P}'} \subseteq \boldsymbol{\mathcal{P}}$ is an n-scale of p. In this case, for any Ramsey function $\mathcal{C}r$, $\mathcal{C}r(p) = {}^{n}/_{m} \cdot \mathcal{C}r(q)$. With respect to ${}^{n}/_{m}$-valued propositions, Ramsey functions cohere with the GRP perfectly.

Essentially, a Ramsey function scales every non-minimal ${}^{n}/_{m}$-valued proposition relative to a maximal proposition, which has a stipulated value. With respect to pairs of propositions that cannot be so scaled, however, a Ramsey function *may* conflict with the GRP. An example where this could occur is:



Where

$$\Omega \succ \{\omega_2, \omega_3\} \succ \{\omega_1, \omega_2\} \sim \{\omega_1, \omega_3\} \succ \{\omega_2\} \sim \{\omega_3\} \succ \{\omega_1\} \succ \varnothing$$

The only non-trivial n-scale is the 2-scale $\{\{\omega_2\}, \{\omega_3\}\}$ of $\{\omega_2, \omega_3\}$; but, since $\{\omega_2, \omega_3\}$ can't be scaled relative to Ω, the values of $\{\omega_2\}, \{\omega_3\}$ and $\{\omega_2, \omega_3\}$ are left indeterminate.

Call any proposition that's ${}^{n}/_{m}$-valued 'Ramsey-scalable'. Ramsey says nothing about how to measure propositions that aren't Ramsey-scalable, and this is an important lacuna in his proposal—though perhaps not a very troubling one. One might assume that such cases don't exist. Let $\mathfrak{N}$ designate the set of Ramsey-scalable propositions. Then, the assumption would be:

C7. $\mathfrak{N} = \mathfrak{B}$.

C7 isn't implied by C1–C6. However, it *is* implied by C5 plus Continuity. (Let q be maximal, $\varnothing$ minimal, and consider any non-minimal p; Continuity then states that $\{p\}$ is a 1-scale of the union of n members of an m-scale of q, which is strictly more than C7 requires.) In other words, probabilistic comparativists have nothing to fear from a condition like C7.

But C7 only ensures that every $p \in \boldsymbol{\mathcal{B}}$ is Ramsey-scalable. It isn't *yet* enough to ground a plausible comparativist story. There's still two additional problems that can arise in the absence of further conditions on $\succsim$.

First: nothing's been said to guarantee that Definition 6 is *consistent*. Without further assumptions, it's entirely possible for, e.g., $p \sim q$, where for some r, p belongs to a 2-scale of r and q belongs to a 3-scale of r. This is clearly undesirable: α can't believe p to the degrees $^1/_2$ and $^1/_3$ simultaneously! If Ramsey functions are to be well-defined, we'll need to ensure that if p is both $^n/_m$-valued and $^{n'}/_{m'}$-valued, then $^n/_m = {}^{n'}/_{m'}$.

Second: nothing's been said to guarantee that a Ramsey function relative to $\succsim$ will even *agree* with $\succsim$. Indeed, nothing ensures that $\mathcal{C}r(p) \geq \mathcal{C}r(q)$ if *or* only if $p \succsim q$—p could be $^1/_2$-valued, and q $^1/_4$-valued, yet $q \succsim p$. This is also undesirable: if the order of the values we assign propositions don't match up to the probability ranking, then there's no natural sense in which those values are a measure of the *strengths* with which those propositions are believed.

In the context of C7, we can kill these two birds with one stone by making the following rather strong assumption:

> **C8.** If p is $^n/_m$-valued and q is $^{n'}/_{m'}$-valued, then $p \succsim q$ iff $^n/_m \geq {}^{n'}/_{m'}$.

C8 is obviously necessary (and given C7, sufficient) to avoid both worries, as the following representation theorem establishes:

> **Theorem 1.** *$\succsim$ satisfies C8 iff there exists a Ramsey function $\mathcal{C}r$ with respect to $\succsim$; furthermore, $\mathcal{C}r$ is the unique Ramsey function relative to $\succsim$ that agrees with $\succsim$ iff $\succsim$ satisfies C7.*

Proof. Existence, left-to-right: (i) Assume C8. If p is $^n/_m$-valued and $^{n'}/_{m'}$-valued, then $^n/_m = {}^{n'}/_{m'}$; so we're able to assign a unique $r \in [0, 1]$ to every $p \in \mathfrak{N}$ and thus define a Ramsey function $\mathcal{C}r$ relative to $\succsim$ on $\mathfrak{N}$. $\mathcal{C}r$ can then obviously be extended to $\mathfrak{B}$. (ii) Suppose for $p, q \in \mathfrak{N}$, $p \succsim q$. Where p is $^n/_m$-valued and q is $^{n'}/_{m'}$-valued, $^n/_m \geq {}^{n'}/_{m'}$. By (i), $\mathcal{C}r(p) \geq \mathcal{C}r(q)$. Next suppose $\mathcal{C}r(p) \geq \mathcal{C}r(q)$. Since $\mathcal{C}r$ is a Ramsey function, p is $^n/_m$-valued and q is $^{n'}/_{m'}$-valued, for $^n/_m \geq {}^{n'}/_{m'}$. So from C8, $p \succsim q$. *Existence, right-to-left* is straightforward and omitted. *Uniqueness*: The left-to-right is obvious. The restriction of $\mathcal{C}r$ to $\mathfrak{N}$ is the unique Ramsey function relative to the restriction of $\succsim$ to $\mathfrak{N}$; so if $\mathfrak{N} = \mathfrak{B}$ then $\mathcal{C}r$ is unique *simpliciter*. □

In light of Theorem 1, let's characterise the precise version of Ramseyan comparativism:

> Precise Ramseyan Comparativism. If $\mathcal{C}r$ is the only Ramsey function relative to α's probability ranking, then $\mathcal{C}r$ is a fully adequate model of α's degrees of belief.

We can now characterise precisely the respects in which precise Ramseyan comparativism is more lenient than probabilistic comparativism.

It's easy to see that C8 is implied already by C1–C6. Indeed, if any probability function $\mathcal{C}r$ agrees with $\succsim$, then $\mathcal{C}r$ is *also* a Ramsey function relative to $\succsim$. Moreover, where C1–C7 hold, then the unique probability function that agrees with $\succsim$ *is* the unique Ramsey function that agrees with $\succsim$. This is important, since it means that precise Ramseyan comparativism is a direct generalisation of any version of precise probabilistic comparativism that makes use of Continuity or any similar condition.

In the other direction, we can also easily see that while C7 and C8 jointly imply C1–C3, they don't imply the much more problematic conditions C4–C6. It's straightforward to find examples where C7 and C8 are satisfied but C4 and C5 aren't; and for an example where C6 is falsified, assume again that $\Omega = \{w_1, w_2, w_3\}$, and:

$$\{w_1, w_2\} \succ \Omega \sim \{w_1\} \sim \{w_2\} \succ \{w_3\} \sim \{w_1, w_3\} \sim \{w_2, w_3\} \sim \varnothing$$

As $\{\{w_1\}, \{w_2\}\}$ is a 2-scale of the maximal $\{w_1, w_2\}$, $\mathcal{C}r(\{w_1, w_2\}) = 1$ and $\mathcal{C}r(\{w_1\}) = 1/2$. Ω isn't maximal, but it's as likely as $\{w_1\}$; so $\mathcal{C}r(\Omega) = 1/2$.

Most of the interesting 'work', as it were, is done by C8. This condition imposes a very limited kind of additive structure on $\succsim$, specifically with respect to probability rankings between propositions constructed out of members of the same n-scale of any n'/m'-valued proposition. Roughly: *within* an n-scale, $\succsim$ behaves probabilistically—but not every proposition is constructible out of the members of an appropriate n-scale, and *across* n-scales $\succsim$ can behave quite irrationally indeed.

5.2 Imprecise Ramseyan comparativism

If we wanted to drop C7 out of the picture, we could take a page out of the imprecise probabilistic comparativist's book and do so by adopting a set-of-functions representation of $\succsim$. For that, we would need to reintroduce at least the conditions C2 and C3. These two conditions are obviously necessary for *any* real-valued function *or* set thereof to agree with $\succsim$, and without C7 it's no longer the case that they're implied by C8.

For simplicity, we focus on the case where $\mathcal{B}$ is countable; thus,

> **Theorem 2.** *Where $\mathcal{B}$ is countable, $\succsim$ satisfies C2, C3 and C8 iff there exists a nonempty set $\mathfrak{F}$ of real-valued functions bounded by 0 and 1 that agrees with $\succsim$, where every $\mathcal{C}r$ in $\mathfrak{F}$ is a Ramsey function relative to $\succsim$. Furthermore, whenever C2, C3 and C8 are satisfied, there will be a unique set $\mathfrak{F}$ that agrees with $\succsim$ that's maximal with respect to inclusion.*

Proof. Existence, left-to-right: Assume C2, C3 and C8, and that $\mathcal{B}$ is countable. We focus on the case where $\mathcal{N} \subset \mathcal{B}$ as C7 trivialises the proof. From C2 and C3, at least one nonempty set $\mathfrak{G}$ of functions $\mathcal{C}r : \mathcal{B} \mapsto \mathbb{R}$ agrees with $\succsim$—see (Evren and Ok 2011, p. 556). We need that there's a nonempty $\mathfrak{G}^* \subseteq \mathfrak{G}$ such that

1. $\mathfrak{G}^*$ agrees with $\succsim$; and
2. $\forall \mathcal{C}r \in \mathfrak{G}^*$, there's a strictly increasing transformation $\mathcal{C}r'$ of $\mathcal{C}r$ such that
   (a) $\mathcal{C}r'$ is bounded by 1 and 0, and
   (b) $\mathcal{C}r'$ is a Ramsey function w.r.t. $\succsim$.

The set $\mathfrak{F}$ of all such transformations will agree with $\succsim$, completing the proof. There are three cases (1) $\mathcal{N}$ is empty; (2) $\mathcal{N}$ contains only the minimal and/or maximal elements of $\mathcal{B}$; and (3) $\mathcal{N}$ contains non-minimal, non-maximal elements of $\mathcal{B}$. The first two are straightforward and omitted. For (3), if $\mathfrak{G}$ agrees with

$\succsim$, and $p \succ q$, then $\mathcal{C}r(p) \geq \mathcal{C}r(q)$, for all $\mathcal{C}r \in \mathfrak{G}$, and $\mathcal{C}r(p) > \mathcal{C}r(q)$, for some $\mathcal{C}r \in \mathfrak{G}$. So for any $\mathcal{C}r \in \mathfrak{G}$, if $p \succ q$ then $\mathcal{C}r(p) > \mathcal{C}r(q)$ or $\mathcal{C}r(p) = \mathcal{C}r(q)$. For $p, q \in \mathfrak{N}$, C8 implies that for any Ramsey function, if $p \succ q$, then $\mathcal{C}r(p) > \mathcal{C}r(q)$; so, it's not in general true that if $\mathfrak{G}$ agrees with $\succsim$, then for every $\mathcal{C}r \in \mathfrak{G}$ there's a strictly increasing transformation of $\mathcal{C}r$ that's also a Ramsey function with respect to $\succsim$. But define $\mathfrak{G}^* \subseteq \mathfrak{G}$ as follows:

$$\mathfrak{G}^\star = \{\mathcal{C}r \in \mathfrak{G} : \text{if } p, q \in \mathfrak{N} \text{ and } p \succ q, \text{ then } \mathcal{C}r(p) > \mathcal{C}r(q)\}.$$

$\mathfrak{G}^\star$ agrees with $\succsim$, and by (ii) above, we know that it's nonempty. Let $\mathfrak{G}^R$ denote the set of restrictions of every $\mathcal{C}r \in \mathfrak{G}^\star$ to $\mathfrak{N}$, now the unique Ramsey function $\mathcal{C}r^R$ on $\mathfrak{N}$ is a strictly increasing transformation of every $\mathcal{C}r \in \mathfrak{G}^R$. So we just have to show that each $\mathcal{C}r \in \mathfrak{G}^\star$ has a strictly increasing transformation that's there's an extension of $\mathcal{C}r^R$ from $\mathfrak{N}$ to $\mathfrak{B}$. Where $\mathfrak{B}$ is countable this is straightforward, given that for every $\mathcal{C}r \in \mathfrak{G}^\star$, $\mathcal{C}r(p)$ is rational. The uniqueness condition is obvious and has been suppressed. □

With Theorem 2 now in place, let's characterise the imprecise variety of 'Ramseyan comparativisms:

> Imprecise Ramseyan Comparativism. If $\mathfrak{F}$ is a non-empty set of Ramsey functions with respect to α's probability ranking, which is maximal with respect to inclusion and agrees with $\succsim$, then $\mathfrak{F}$ is an R-to-L adequate model of α's degrees of belief.

Note that imprecise Ramseyan comparativism only claims R-to-L adequacy. This is because (as we've seen) C2, C3 and C8 are not sufficient for *total* coherence with the GRP. This is a limitation with the theory, but perhaps not a devastating one. In effect, R-to-L adequacy says that we won't go wrong whenever we read cardinal information off of the numbers, though there *may* be some interesting cardinal properties to one's degrees of belief that aren't appropriately captured by their numerical representation. I suspect that many comparativists would be satisfied by this result—after all, nobody said that our numerical representations had to be perfect.

Imprecise Ramseyan comparativism also agrees exactly with (precise and imprecise) probabilistic comparativism whenever C1–C7 are satisfied. We've already shown that this is so for precise Ramseyan comparativism, but it is not so obvious in the case of imprecise Ramseyan comparativism. Nevertheless, consider: if C1–C7, then the probability function $\mathcal{C}r$ that agrees with $\succsim$ is the Ramsey function that agrees with $\succsim$; from imprecise Ramseyan comparativism, $\mathcal{C}r$ is R-to-L adequate, so $\mathcal{C}r$ determines a unique ratio comparison for every pair of non-minimal propositions; and finally, α cannot believe p $^n/_m$ times as much as q and $^{n'}/_{m'}$ as much as q, for $^n/_m \neq {}^{n'}/_{m'}$.

5.3 The importance of C8

Importantly, we can show that C2, C3, and C8 are individually *necessary* for coherence with the GRP. For C2 and C3 this is trivial, for the reasons already mentioned. The more interesting result concerns C8. Given some minimal scaling assumptions, violations of C8 imply that any $\mathcal{C}r$ that agrees with $\succsim$ won't cohere with the GRP:

Theorem 3. *If (i) $\mathcal{C}r$ agrees with $\succsim$, (ii) there are p, q such that $p \succ q$, and (iii) $\mathcal{C}r(r) = 0$ whenever r is minimal, then $\mathcal{C}r$ coheres with the GRP only if C8 is satisfied.*

Proof. Suppose $\succsim$ violates C8 and $\mathcal{C}r$ agrees with $\succsim$. So there exist p, q s.t. p is $^n/_m$-valued, q is $^{n'}/_{m'}$-valued, and not: $(p \succsim q) \leftrightarrow (^n/_m \geq {}^{n'}/_{m'})$. There are three cases: (1) neither p nor q is minimal; (2) both p and q are minimal; or (3) exactly one of p or q is minimal.

Start with case (1). Focus on p, and let max designate some maximal proposition. (If p is $^n/_m$-valued, non-minimal, then max exists.) p is either (i) the union of n members of an m-scale of max, or (ii) the union of n'' members of an m''-scale of … the union of n''' members of an m'''-scale of max. If (i), $\mathcal{C}r$ coheres with the GRP only if $\mathcal{C}r(p) = {}^n/_m \cdot \mathcal{C}r(r)$; if (ii), only if $\mathcal{C}r(p) = {}^{(n'' \cdot \ldots \cdot n''')}/_{(m'' \cdot \ldots \cdot m''')} \cdot \mathcal{C}r(r)$, where $^{(n' \cdot \ldots \cdot n'')}/_{(m' \cdot \ldots \cdot m'')} = {}^n/_m$. The same applies to q, *mutatis mutandis*, so $\mathcal{C}r$ coheres with the GRP only if $\mathcal{C}r(p) = {}^{n'}/_{m'} \cdot \mathcal{C}r(r)$. Assume for *reductio* that $\mathcal{C}r$ coheres with the GRP. Now suppose $^n/_m \geq {}^{n'}/_{m'}$, so $\mathcal{C}r(p) \geq \mathcal{C}r(q)$, and hence $p \succsim q$. In the other direction, suppose $p \succsim q$; so $\mathcal{C}r(p) \geq \mathcal{C}r(q)$, and $^n/_m \geq {}^{n'}/_{m'}$. So, $p \succsim q \leftrightarrow {}^n/_m \geq {}^{n'}/_{m'}$, which violates our assumptions.

Now case (2). Assume there are $p, q \in \mathfrak{B}$ such that $p \succ q$, and that if p is minimal, then $\mathcal{C}r(p) = 0$. If p, q are minimal then $p \sim q$, and if $\mathcal{C}r$ agrees with $\succsim$ then $\mathcal{C}r(p) = \mathcal{C}r(q) > \mathcal{C}r(s)$, for any s such that $s \not\sim p$ (and hence $s \succ p$). Since p, q are $^0/_m$-valued by definition, C8 is violated only if p or q is also $^n/_m$-valued, for $n > 0$. Suppose this of p; then by the earlier reasoning, $\mathcal{C}r$ coheres with the GRP only if $\mathcal{C}r(p) = {}^n/_m \cdot \mathcal{C}r(r)$. Since $^n/_m > 0$ and $\mathcal{C}r(r) > 0$, this is false; so $\mathcal{C}r$ conflicts with the GRP. Case (3) is then straightforward, and the proof of the corollary (for sets of functions) stated just below follows the same structure. Both proofs are omitted. □

Corollary: under the same assumptions, *mutatis mutandis*, any set of real-valued functions $\mathfrak{F}$ will cohere with the GRP only if C8 is satisfied.

In other words, assuming just that $\succsim$ is non-trivial and that minimal propositions can be assigned value 0, any kind of coherence with the GRP implies C2, C3 and C8. Thus we have a set of minimal conditions necessary for the union of pseudodisjoint sets to behave like addition with respect to $\succsim$.

6. Conclusion

Let's take stock. The standard comparativist strategy for explaining cardinality is based on a purported analogy with the measurement of certain extensive quantities like length or mass: to say that p is n times more likely than q, we just need to be able to say that p is as likely as the union of n 'duplicates' of q, where the 'duplicates' are propositions that are equiprobable and pairwise (pseudo)disjoint. The two Ramseyan comparativisms just outlined offer an accounts of when this kind of 'adding' is meaningful that generalise the conditions assumed by probabilistic comparativism, applying to a wide range of probability rankings that aren't probabilistically representable. In particular, we've shown that comparativists can in principle do without any appeal to C4–C6, and can avoid the problems that those conditions bring in their wake.

Moreover, we know that the union of pseudodisjoint sets behaves like addition only given C2, C3 and C8. Inasmuch as comparativists want to retain the analogy with the measurement and mass as it's usually understood—i.e., in terms of the union of either disjoint or pseudodisjoint propositions—Ramseyan comparativism is about as general as it gets.

This is a result of some importance for comparativism, but let me be clear about exactly what has been achieved here. It remains to be seen whether it's *correct* to say that p is n times more likely than q if and only if p is as likely as the union of n pseudodisjoint duplicates of q. This is not a question that I've considered in this paper, and it raises a host of tricky issues that need careful consideration. But we now have the resources necessary to address that question. We know what it takes for the analogy with mass to hold, so we can ask: are those three conditions, C2, C3 and C8, plausibly satisfied by actual agents—or at least, by the kinds of agents whom we are happy to say have numerical degrees of belief that carry interesting cardinal information? And, if so, does the GRP correctly predict our considered judgements about the degrees of beliefs of such agents? A positive answer to both of these questions would be a strong win in favour of comparativism.[11]

Appendix: Conditional Ramseyan comparativism

The way I developed Ramsey's idea in §5 is not the only way that it could have been developed. An alternative way to gloss Ramsey's thought would be:

> α believes p to degree $2/3$ just in case she considers p to be just as likely as one of two tickets winning, in a fair lottery with exactly three tickets.

Since one can always *imagine* holding n tickets in a fair lottery with m tickets overall, it ought to be possible to use this kind of comparison to determine the probability of any proposition to within some fairly narrow range. However, what if α is certain that there are no such lotteries?

Even if α is sure that she holds no tickets to any fair lotteries, we can still ask what her to judge the relative probabilities of winning *under the supposition* that she has n tickets in a fair m-ticket lottery. With this as our starting point, we might want to develop Ramsey's idea into a version of comparativism that, like the theories of Koopman (1940b,a) and more recently Hawthorne (2016), takes α's *conditional* probability rankings as basic.

Hence, let α's conditional probability ranking be modelled by a binary relation $\succsim^q$ on $\boldsymbol{\mathcal{B}} \times \boldsymbol{\mathcal{B}}$, where

$(p, q) \succsim^q (r, s)$ iff α believes p given q at least as much as she believes r given s.

We can then construct conditional versions of the Ramseyan theories outlined in §5.

Definition 7. A set of n propositions $\boldsymbol{\mathcal{P}} = \{p_1, ..., p_n\}$ is a *conditional n-scale* iff, for some minimal pair (q, r),

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(i) $(p_i, \bigcup \boldsymbol{\mathcal{P}}) \sim^q (p_j, \bigcup \boldsymbol{\mathcal{P}}) \succ^q (q, r)$, for $i, j = 1, \ldots, n$, and
(ii) If $\boldsymbol{\mathcal{P}}' \subseteq \boldsymbol{\mathcal{P}}$ and $|\boldsymbol{\mathcal{P}}'| \geq 2$, then $(\bigcap \boldsymbol{\mathcal{P}}', \bigcup \boldsymbol{\mathcal{P}}) \sim^q (q, r)$.

That is, a set of propositions $\boldsymbol{\mathcal{P}}$ is a conditional n-scale just in case, under the supposition that at least one of its members must be true, the intersection of any two or more of its members is considered wholly unlikely, and each is just as likely as any other (and not wholly likely).

Definition 7 is a very light generalisation of Koopman's (1940a) definition of an 'n-scale', and it is the natural correlate of Definition 4 in the context of conditional comparativism. With Definition 7 in place, we can restate the GRP in the new context:

Conditional Ratio Principle. For integers n, m, k such that $n \leq m \leq k$, if there exists a conditional k-scale $\boldsymbol{\mathcal{P}} = \{p_1, \ldots, p_n, \ldots, p_m, \ldots, p_k\}$ s.t.:

(i) $(p, q) \sim (p_1 \cup \cdots \cup p_n, \bigcup \boldsymbol{\mathcal{P}})$ and
(ii) $(r, s) \sim (p_1 \cup \cdots \cup p_m, \bigcup \boldsymbol{\mathcal{P}})$,

then (p, q) is believed $^n/_m$ times as much as (r, s).

Likewise, we can state the obvious correlates for Definition 5 and Definition 6:

Definition 8. For integers n, m such that $m \geq n \geq 0$, $m > 0$, a pair of propositions (p, q) is

(i) $^0/_m$*-valued* if (p, q) is minimal and $^m/_m$*-valued* if (p, q) is maximal, and
(ii) $^n/_m$*-valued* if there exists an m-scale $\boldsymbol{\mathcal{P}} = \{p_1, \ldots, p_m\}$ such that (p, q) $\sim^q (p_1 \cup \cdots \cup p_n, \bigcup \boldsymbol{\mathcal{P}})$.

Definition 9. $\mathcal{C}r : \mathfrak{B} \times \mathfrak{B} \mapsto [0, 1]$ is a *conditional Ramsey function* (relative to $\succsim$) iff, for all $(p, q) \in \mathfrak{B} \times \mathfrak{B}$, if (p, q) is $^n/_m$-valued, then $\mathcal{C}r(p, q) = {}^n/_m$.

And, finally,

Theorem 4. *If $\succsim^q$ is a binary relation on a countable set $\mathfrak{B} \times \mathfrak{B}$, $\mathfrak{B} \subseteq \wp(\Omega)$, then there exists a non-empty set $\mathfrak{F}$ of functions into $[0, 1]$ that agrees with $\succsim^q$, where every $\mathcal{C}r \in \mathfrak{F}$ is a conditional Ramsey function relative to $\succsim$, iff $\succsim^q$ satisfies:*

C9. *$\succsim^q$ is reflexive.*
C10. *$\succsim^q$ is transitive.*
C11. *If (p, q) is $^n/_m$-valued and (r, s) is $^{n'}/_{m'}$-valued, then $(p, q) \succsim (r, s)$ iff $^n/_m \geq {}^{n'}/_{m'}$.*

The proof of Theorem 4 is essentially the same as that of Theorem 2, and has been ommitted. It should go without saying that axioms C9–C11 are weaker than those usually required for there to exist a (set of) conditional probability function(s) $\mathcal{C}r(\cdot|\cdot)$ that agrees with $\succsim^q$ (see Koopman 1940a, Hawthorne 2016, for details).

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